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Alexander J. Zaslavski

Algorithms for Solving Common Fixed Point Problems

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Algorithms for Solving Common Fixed Point Problems

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Preface

In this book, we study approximate solutions of common fixed point and convex feasibility problems in the presence of perturbations. A convex feasibility problem is to find a point which belongs to the intersection of a given finite family of subsets of a Hilbert space. This problem is a special case of a common fixed point problem which is to find a common fixed point of a finite family of self-mappings of a Hilbert space. The study of these problems has recently been a rapidly growing area of research. This is due not only to theoretical achievements in this area, but also because of numerous applications to engineering and, in particular, to computed tomography and radiation therapy planning. In the book, we consider a number of algorithms, which are known as important tools for solving convex feasibility and common fixed point problems. According to the results known in the literature, these algorithms should converge to a solution. But it is clear that in practice it is sufficient to find a good approximate solution instead of constructing a minimizing sequence. In our recent book *Approximate Solutions of Common Fixed Point Problems*, Springer, 2016, we analyzed these algorithms and showed that almost all exact iterates generated by them are approximate solutions. Moreover, we obtained an estimate of the number of iterates which are not approximate solutions. This estimate depends on the algorithm but does not depend on the starting point. In this book, our first goal is to generalize these results for perturbed algorithms in the case when perturbations are summable. These generalizations are important because such results find interesting applications and are important ingredients in superiorization and perturbation resilience of algorithms. The superiorization methodology works by taking an iterative algorithm, investigating its perturbation resilience, and then using proactively such perturbations in order to "force" the perturbed algorithm to do in addition to its original task something useful. Our second goal is to study approximate solutions of common fixed point problems in the presence of perturbations which are not necessarily summable. Note that in our recent book mentioned earlier it was shown that if perturbations are small enough, then we have an approximate solution during a certain number of iterates, and an estimate for this number of iterates was obtained. But these results do not show what happens with subsequent iterates, when an approximated solution is obtained. In this book, we show that if our algorithms are cyclic and a computational error is sufficiently small, then beginning from a certain instant of time iterates become approximate solutions. This instant of time depends on the algorithm but does not depend on its starting point.

This book contains eight chapters. Chapter 1 is an introduction. In Chapter 2, we study iterative methods in metric spaces. The dynamic string-averaging methods for common fixed point problems in normed space are analyzed in Chapter 3. Dynamic string methods, for common fixed point problems in a metric space, are introduced and studied in Chapter 4. Chapter 5 is devoted to the study of the convergence of an abstract version of the algorithm which is called in the literature as component-averaged row projections or CARP. In Chapter 6, we study a proximal algorithm for finding a common zero of a family of maximal monotone operators. In Chapter 7, we extend the results of Chapter 6 for a dynamic string-averaging version of the proximal algorithm. In Chapter 8, subgradient projection algorithms for convex feasibility problems are studied for infinite-dimensional Hilbert spaces.

Theorems 2.1 and 3.1 were obtained in [125]. All other results are new.

Haifa, Israel Alexander J. Zaslavski
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Chapter 1
Introduction

In this book we study approximate solutions of common fixed point and convex feasibility problems in the presence of perturbations. A convex feasibility problem is to find a point which belongs to the intersection of a given finite family of convex subsets of a Hilbert space. This problem is a special case of a common fixed point problem which is to find a common fixed point of a finite family of nonlinear mappings in a Hilbert space. Our goal is to show the convergence of algorithms, which are known as important tools for solving convex feasibility and common fixed point problems. Some of these algorithms are discussed is this chapter.

1.1 Common Fixed Point Problems in a Metric Space

Let (X, d) be a metric space. For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{d(x, y) : \ y \in E\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ d(x, y) \leq r\}.$$

Let m be a natural number, $\bar{c} \in (0, 1)$ and let $P_i : X \to X$, $i = 1, \dots, m$ be self-mappings of the space X. Suppose that for every $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \neq \emptyset.$$

We also suppose that for every $i \in \{1, \dots, m\}$ the inequality

$$d(z, x)^2 \geq d(z, P_i(x))^2 + \bar{c} d(x, P_i(x))^2$$

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holds for all $x \in X$ and all $z \in \mathrm{Fix}(P_i)$. Set

$$F = \cap_{i=1}^m \mathrm{Fix}(P_i).$$

Elements of the set F are solutions of the common fixed point problem.

It should be mentioned that if the space X is Hilbert and for all $i = 1, \dots, m$, the mapping P_i is the projection on a convex closed set $C_i \subset X$, then we have a convex feasibility problem which has numerous applications to engineering and, in particular, to computed tomography and radiation therapy planning.

For every $\epsilon > 0$ and every $i \in \{1, \dots, m\}$ set

$$F_\epsilon(P_i) = \{x \in X : \; d(x, P_i(x)) \le \epsilon\},$$

$$\tilde{F}_\epsilon(P_i) = \{y \in X : \; d(y, F_\epsilon(P_i)) \le \epsilon\},$$

$$F_\epsilon = \cap_{i=1}^m F_\epsilon(P_i),$$

$$\tilde{F}_\epsilon = \cap_{i=1}^m \tilde{F}_\epsilon(P_i).$$

Elements of F_ϵ ($\tilde{F}_\epsilon$ respectively) are considered as ϵ-approximate solutions of the common fixed point problem.

Fix $\theta \in X$ and a natural number $\bar{N} \ge m$.

Denote by $\mathcal{R}$ the set of all mappings $r : \{1, 2, \dots\} \to \{1, \dots, m\}$ such that for each number j,

$$\{1, \dots, m\} \subset \{r(j), \dots, r(j + \bar{N} - 1)\}.$$

Every $r \in \mathcal{R}$ generates the following algorithm.

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration point x_k calculate the next iteration point x_{k+1} by

$$x_{k+1} = P_{r(k+1)}(x_k).$$

Denote by $\mathrm{Card}(A)$ the cardinality of a set A. The following result is presented in Chapter 3 of [124]. It was obtained in [123].

Theorem 1.1 *Let* $M > 0$ *satisfy*

$$B(\theta, M) \cap F \ne \emptyset,$$

$\epsilon > 0$,

$$r \in \mathcal{R},$$

$$x_0 \in B(\theta, M)$$

and let $\{x_i\}_{i=1}^{\infty} \subset X$ *satisfy for each natural number* i*,*

$$x_i = P_{r(i)}(x_{i-1}).$$

Then

$$Card(\{i \in \{0, 1, \dots\} : \ x_i \notin \tilde{F}_\epsilon\}) \le 4\bar{N}^3 M^2 \bar{c}^{-1} \epsilon^{-2}.$$

This result shows that almost all exact iterates generated by our algorithms are ϵ-approximate solutions. Moreover, it provides us the estimate of the number of iterates which are not ϵ-approximate solutions.

One of our goals is to generalize this result for perturbed algorithms in the case when perturbations are summable. Such generalizations are important because they find interesting applications and are essential ingredients in superiorization and perturbation resilience of algorithms. See [9, 29, 45, 47, 49, 54, 75] and the references mentioned therein. The superiorization methodology works by taking an iterative algorithm, investigating its perturbation resilience, and then using proactively such perturbations in order to "force" the perturbed algorithm to do in addition to its original task something useful. This methodology can be explained by the following result on convergence of inexact iterates.

Assume that $(Z, \|\cdot\|)$ is a Banach space, $T : Z \to Z$ is nonexpansive mapping such that

$$\|T(x) - T(y)\| \le \|x - y\| \text{ for all } x, y \in Z,$$

for each $x \in Z$, the sequence $\{T^n(x)\}_{n=1}^{\infty}$ converges in the norm topology. $x_0 \in Z$, $\{\beta_k\}_{k=0}^{\infty}$ is a sequence of positive numbers satisfying

$$\sum_{k=0}^{\infty} \beta_k < \infty, \tag{1.1}$$

$\{v_k\}_{k=0}^{\infty} \subset Z$ is a norm bounded sequence and that for any integer $k \ge 0$,

$$x_{k+1} = T(x_k + \beta_k v_k). \tag{1.2}$$

Then it follows from the result of [27] that the sequence $\{x_k\}_{k=0}^{\infty}$ converges in the norm topology of Z and its limit is a fixed point of T. In this case the mapping T is called bounded perturbations resilient (see [49] and Definition 10 of [45]). In other words, if exact iterates of a nonexpansive mapping converge, then its inexact iterates with bounded summable perturbations converge too. Now assume that $x_0 \in X$ and the sequence $\{\beta_k\}_{k=0}^{\infty}$ satisfying (1.1) are given and we need to find an approximate fixed point of T. In order to meet this goal we construct a sequence $\{x_k\}_{k=1}^{\infty}$ defined by (1.2). Under an appropriate choice of the bounded sequence $\{v_k\}_{k=0}^{\infty}$, the sequence $\{x_k\}_{k=1}^{\infty}$ possesses some useful property. For example, the sequence $\{f(x_k)\}_{k=1}^{\infty}$ can be decreasing, where f is a given function.

In the next chapter we prove the following result which was obtained in [125].

Theorem 1.2 *Assume that* $M > 0$ *satisfies*

$$B(\theta, M) \cap F \neq \emptyset,$$

ϵ *is a positive number and that a sequence* $\{\epsilon_i\}_{i=1}^{\infty} \subset [0, \infty)$ *satisfies*

$$\Lambda := \sum_{i=1}^{\infty} \epsilon_i < \infty.$$

Let a natural number n_0 *be such that for all integers* $i \geq n_0$,

$$\epsilon_i < (2\bar{N})^{-1}\epsilon.$$

Let

$$r \in \mathcal{R},$$

$$x_0 \in B(\theta, M)$$

and let a sequence $\{x_i\}_{i=1}^{\infty} \subset X$ *satisfy for each natural number* i,

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \epsilon_i.$$

Then

$$Card(\{i \in \{0, 1, \ldots\} : \; x_i \notin \tilde{F}_\epsilon\})$$

$$\leq n_0 + 4\bar{N}^3\bar{c}^{-1}\epsilon^{-2}((2M + \Lambda)^2 + 2\Lambda(2M + \Lambda)).$$

Denote by $\mathcal{R}_{per}$ the set of all $r \in \mathcal{R}$ such that for each integer $i \geq 1$,

$$r(i + \bar{N}) = r(i).$$

The next result, which will be proved in Chapter 2, establishes, for every positive number ϵ, the convergence of cyclic algorithms to the set of approximate fixed points F_ϵ.

Theorem 1.3 *Assume that for every* $i \in \{1, \ldots, m\}$ *and every pair of points* $x, y \in X$,

$$d(P_i(x), P_i(y)) \leq d(x, y).$$

Suppose that $M_* > 1$ *and that the following property holds:*

(P1) for each $\delta > 0$ *there exists* $z_\delta \in B(\theta, M_*)$ *such that*

$$B(z_\delta, \delta) \cap Fix(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m.$$

Let $M \geq M_*$, $\epsilon > 0$,

$$r \in \mathcal{R}_{per},$$

$$x_0 \in B(\theta, M),$$

$\{x_i\}_{i=1}^{\infty} \subset X$ *and let for each natural number* i,

$$x_i = P_{r(i)}(x_{i-1}).$$

Then for every integer $i \geq \bar{N}(64M^4\bar{c}^{-3}\bar{N}^2\epsilon^{-4}(2\bar{N}+1)^2+1)$,

$$x_i \in F_\epsilon.$$

Using this theorem, in Chapter 2 we establish the next convergence results under the presence of small perturbations.

For a real number $z \in R^1$ we set $\lfloor z \rfloor = \max\{i : \ i \text{ is an integer and } i \leq z\}$.

Theorem 1.4 *Assume that for every* $i \in \{1, \dots, m\}$ *and every pair of points* $x, y \in X$,

$$d(P_i(x), P_i(y)) \leq d(x, y).$$

Suppose that $M_* > 1$, *property (P1) holds,* $\tilde{M} > M_*$, $r_0 > 0$ *and*

$$F_{r_0} \subset B(\theta, \tilde{M}).$$

Let $M \geq \tilde{M}$, $\epsilon \in (0, r_0]$,

$$q_0 = 3 + \lfloor 4^7 M^4 \bar{N}^2 (2\bar{N}+1)^4 \bar{c}^{-3} \epsilon^{-4} \rfloor,$$

a positive number δ *satisfy*

$$2\delta q_0 \bar{N} \leq \epsilon/4,$$

$$r \in \mathcal{R}_{per},$$

$$x_0 \in B(\theta, M)$$

and let $\{x_i\}_{i=1}^{\infty} \subset X$ *satisfy for each natural number* i,

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \delta.$$

Then for every integer $i \geq q_0\bar{N}$,

$$x_i \in F_\epsilon.$$

1.2 Common Fixed Point Problems in a Hilbert Space

In Chapter 3 we study the convergence of dynamic string-averaging methods which were first introduced by Y. Censor, T. Elfving, and G. T. Herman in [48] for solving a convex feasibility problem, when a given collection of sets is divided into blocks and the algorithms operate in such a manner that all the blocks are processed in parallel. Iterative methods for solving common fixed point problems is a special case of dynamic string-averaging methods with only one block. Iterative methods and dynamic string-averaging methods are important tools for solving common fixed point problems in a Hilbert space [1, 3, 6–9, 11, 12, 14, 17, 18, 24, 26, 28–30, 35, 38, 39, 41–46, 48, 49, 51–53, 56–58, 60–62, 65, 68, 69, 89, 96–98, 106, 107, 110, 112, 114, 117–122].

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ which induces a complete norm $\|\cdot\|$.

For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{\|x - y\| : \ y \in E\}.$$

For every point $x \in X$ and every positive number $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \leq r\}.$$

Suppose that m is a natural number, $\bar{c} \in (0, 1)$, $P_i : X \to X$, $i = 1, \dots, m$, for every integer $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \neq \emptyset$$

and that the inequality

$$\|z - x\|^2 \geq \|z - P_i(x)\|^2 + \bar{c}\|x - P_i(x)\|^2$$

holds for every integer $i \in \{1, \dots, m\}$, every point $x \in X$, and every point $z \in \mathrm{Fix}(P_i)$. Set

$$F = \cap_{i=1}^m \mathrm{Fix}(P_i).$$

For every positive number ϵ and every integer $i \in \{1, \dots, m\}$ set

$$F_\epsilon(P_i) = \{x \in X : \|x - P_i(x)\| \le \epsilon\},$$

$$\tilde{F}_\epsilon(P_i) = F_\epsilon(P_i) + B(0, \epsilon),$$

$$F_\epsilon = \cap_{i=1}^m F_\epsilon(P_i)$$

and

$$\tilde{F}_\epsilon = \cap_{i=1}^m \tilde{F}_\epsilon(P_i)$$

A point belonging to the set F is a solution of our common fixed point problem while a point which belongs to the set $\tilde{F}_\epsilon$ is its ϵ-approximate solution.

In Chapter 3 we obtain a good approximative solution of the common fixed point problem applying a dynamic string-averaging method with variable strings and weights which is described below.

By an index vector, we mean a vector $t = (t_1, \ldots, t_p)$ such that $t_i \in \{1, \ldots, m\}$ for all $i = 1, \ldots, p$.

For an index vector $t = (t_1, \ldots, t_q)$ set

$$p(t) = q, \;\; P[t] = P_{t_q} \cdots P_{t_1}.$$

It is not difficult to see that for each index vector t

$$P[t](x) = x \text{ for all } x \in F,$$

$$\|P[t](x) - P[t](y)\| = \|x - P[t](y)\| \le \|x - y\|$$

for every point $x \in F$ and every point $y \in X$.

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ satisfies } \sum_{t \in \Omega} w(t) = 1.$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{t \in \Omega} w(t) P[t](x), \;\; x \in X.$$

It is easy to see that

$$P_{\Omega,w}(x) = x \text{ for all } x \in F,$$

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| = \|x - P_{\Omega,w}(y)\| \le \|x - y\|$$

for every point $x \in F$ and every point $y \in X$.

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm.

Initialization: select an arbitrary point $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k).$$

Fix a number

$$\Delta \in (0, m^{-1}]$$

and an integer

$$\bar{q} \geq m.$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \leq \bar{q} \text{ for all } t \in \Omega,$$

$$w(t) \geq \Delta \text{ for all } t \in \Omega.$$

Fix a natural number $\bar{N}$.

In the studies of the common fixed point problem the goal is to find a point $x \in F$. In order to meet this goal we apply an algorithm generated by

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

such that for each natural number j,

$$\{1, \ldots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \ldots, t_{p(t)}\}).$$

This algorithm generates, for any starting point $x_0 \in X$, a sequence $\{x_k\}_{k=0}^{\infty} \subset X$, where

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k).$$

In Chapter 2 of [124] we obtained the following result which shows that almost all exact iterates generated by our algorithms are ϵ-approximate solutions. Moreover, it provides us the estimate of the number of iterates which are not ϵ-approximate solutions.

Theorem 1.5 *Let* $M > 0$ *satisfy*

$$B(0, M) \cap F \neq \emptyset$$

and let $\epsilon > 0$. *Assume that*

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for every natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}),$$

$$x_0 \in B(0, M)$$

and $\{x_i\}_{i=1}^{\infty} \subset X$ *satisfies for every natural number* i,

$$x_i = P_{\Omega_i, w_i}(x_{i-1}).$$

Then

$$Card(\{i \in \{0, 1, \dots\} : \ x_i \notin \tilde{F}_\epsilon\}) \le \bar{N}(4M^2\bar{c}^{-1}\Delta^{-1}\epsilon^{-2}(\bar{N}+1)^2\bar{q}^2 + 1) + 1.$$

In Chapter 3 of this book we generalize this result for perturbed algorithms in the case when perturbations are summable. We also establish, for every positive number ϵ, the convergence of cyclic algorithms to the set of approximate fixed points F_ϵ in the presence of small computational errors.

1.3 Proximal Point Algorithm

Proximal point method is an important tool in solving optimization problems [4, 40, 55, 72, 78, 81, 90, 100, 111]. It is also used for solving variational inequalities with monotone operators [2, 10, 13, 15, 16, 19–23, 25, 76, 80, 84–86, 101, 104, 105] which is an important topic of nonlinear analysis and optimization [31–34, 36, 37, 50, 59, 64, 66, 67, 70, 71, 73, 74, 77, 79, 82, 83, 87, 88, 91, 102, 103, 108, 109, 113, 115, 116]. In Chapter 6 we study the convergence of an iterative proximal point method to a common zero of a finite family of maximal monotone operators in a Hilbert space, under the presence of perturbations.

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ which induces the norm $\| \cdot \|$.

A multifunction $T : X \to 2^X$ is called a monotone operator if and only if

$$\langle z - z', w - w' \rangle \ \ge 0 \quad \forall z, z', w, w' \in X$$

$$\text{such that } w \in T(z) \text{ and } w' \in T(z').$$

It is called maximal monotone if, in addition, the graph

$$\{(z, w) \in X \times X : \ w \in T(z)\}$$

is not properly contained in the graph of any other monotone operator $T' : X \to 2^X$. A fundamental problem consists in determining an element z such that $0 \in T(z)$. For example, if T is the subdifferential ∂f of a lower semicontinuous convex function $f : X \to (-\infty, \infty]$, which is not identically infinity, then T is maximal monotone (see [93, 95]), and the relation $0 \in T(z)$ means that z is a minimizer of f.

Let $T : X \to 2^X$ be a maximal monotone operator. The proximal point algorithm generates, for any given sequence of positive real numbers and any starting point in the space, a sequence of points and the goal is to show the convergence of this sequence. Note that in a general infinite-dimensional Hilbert space this convergence is usually weak. The proximal algorithm for solving the inclusion $0 \in T(z)$ is based on the fact established by Minty [92], who showed that, for each $z \in X$ and each $c > 0$, there is a unique $u \in X$ such that

$$z \in (I + cT)(u),$$

where $I : X \to X$ is the identity operator ($Ix = x$ for all $x \in X$).

The operator

$$P_{c,T} := (I + cT)^{-1}$$

is therefore single-valued from all of X onto X (where c is any positive number). It is also nonexpansive:

$$\|P_{c,T}(z) - P_{c,T}(z')\| \le \|z - z'\| \text{ for all } z, z' \in X$$

and

$$P_{c,T}(z) = z \text{ if and only if } 0 \in T(z).$$

Following the terminology of Moreau [95] $P_{c,T}$ is called the proximal mapping associated with cT.

The proximal point algorithm generates, for any given sequence $\{c_k\}_{k=0}^{\infty}$ of positive real numbers and any starting point $z^0 \in X$, a sequence $\{z^k\}_{k=0}^{\infty} \subset X$, where

$$z^{k+1} := P_{c_k,T}(z^k), \ k = 0, 1, \ldots$$

It is not difficult to see that the

$$\mathrm{graph}(T) := \{(x, w) \in X \times X : \ w \in T(x)\}$$

is closed in the norm topology of $X \times X$.

Set

$$F(T) = \{z \in X : \ 0 \in T(z)\}.$$

Usually algorithms considering in the literature generate sequences which converge weakly to an element of $F(T)$. In Chapter 6, for a given $\epsilon > 0$, we are interested to find a point x for which there is $y \in T(x)$ such that $\|y\| \leq \epsilon$. This point x is considered as an ϵ-approximate solution.

For every point $x \in X$ and every nonempty set $A \subset X$ define

$$d(x, A) := \inf\{\|x - y\| : \ y \in A\}.$$

For every point $x \in X$ and every positive number r put

$$B(x, r) = \{y \in X : \ \|x - y\| \leq r\}.$$

We denote by $\mathrm{Card}(A)$ the cardinality of the set A.

We apply the proximal point algorithm in order to obtain a good approximation of a point which is a common zero of a finite family of maximal monotone operators and a common fixed point of a finite family of quasi-nonexpansive operators.

Let $\mathcal{L}_1$ be a finite set of maximal monotone operators $T : X \to 2^X$ and $\mathcal{L}_2$ be a finite set of mappings $T : X \to X$. We suppose that the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is nonempty. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

Let $\bar{c} \in (0, 1]$ and let $\bar{c} = 1$, if $\mathcal{L}_2 = \emptyset$.

We suppose that

$$F(T) \neq \emptyset \text{ for any } T \in \mathcal{L}_1$$

and that for every mapping $T \in \mathcal{L}_2$,

$$\mathrm{Fix}(T) := \{z \in X : \ T(z) = z\} \neq \emptyset,$$

$$\|z - x\|^2 \geq \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2$$

$$\text{for all } x \in X \text{ and all } z \in \mathrm{Fix}(T).$$

Let $\bar{\lambda} > 0$ and let $\bar{\lambda} = \infty$ and $\bar{\lambda}^{-1} = 0$, if $\mathcal{L}_1 = \emptyset$. Let a natural number

$$l \geq \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2).$$

Denote by $\mathcal{R}$ the set of all mappings

$$S : \{0, 1, 2, \dots\} \to \mathcal{L}_2 \cup \{P_{c,T} : \ T \in \mathcal{L}_1, \ c \in [\bar{\lambda}, \infty)\}$$

such that the following properties hold:

(P1) for every nonnegative integer p and every mapping $T \in \mathcal{L}_2$ there exists an integer $i \in \{p, \dots, p+l-1\}$ satisfying $S(i) = T$;

(P2) for every nonnegative integer p and every monotone operator $T \in \mathcal{L}_1$ there exist an integer $i \in \{p, \dots, p+l-1\}$ and a number $c \geq \bar{\lambda}$ satisfying that $S(i) = P_{c,T}$.

Suppose that

$$F := (\cap_{T\in\mathcal{L}_1} F(T)) \cap (\cap_{Q\in\mathcal{L}_2} \mathrm{Fix}(Q)) \neq \emptyset.$$

Let $\epsilon > 0$. For every monotone operator $T \in \mathcal{L}_1$ define

$$F_\epsilon(T) = \{x \in X : \ T(x) \cap B(0, \epsilon) \neq \emptyset\}$$

and for every mapping $T \in \mathcal{L}_2$ set

$$\mathrm{Fix}_\epsilon(T) = \{x \in X : \ \|T(x) - x\| \leq \epsilon\}.$$

Define

$$F_\epsilon = (\cap_{T\in\mathcal{L}_1} F_\epsilon(T)) \cap (\cap_{Q\in\mathcal{L}_2} \mathrm{Fix}_\epsilon(Q)),$$

$$\tilde{F}_\epsilon = (\cap_{T\in\mathcal{L}_1} \{x \in X : \ d(x, F_\epsilon(T)) \leq \epsilon\})$$

$$\cap(\cap_{Q\in\mathcal{L}_2} \{x \in X : \ d(x, \mathrm{Fix}_\epsilon(Q)) \leq \epsilon\}).$$

We are interested to find solutions of the inclusion $x \in F$. In order to meet this goal we apply algorithms generated by mappings $S \in \mathcal{R}$. More precisely, we associate with every mapping $S \in \mathcal{R}$ the algorithm which generates, for every starting point $x_0 \in X$, a sequence of points $\{x_k\}_{k=0}^\infty \subset X$ such that

$$x_{k+1} := [S(k)](x_k), \ k = 0, 1, \dots.$$

In Chapter 8 of [124] we obtained the following result which shows that almost all exact iterates, generated by our algorithms are ϵ-approximate solutions. Moreover, it provides us the estimate of the number of iterates which are not ϵ-approximate solutions.

Theorem 1.6 *Let $M > 0$, $\epsilon > 0$,*

$$B(0, M) \cap F \neq \emptyset.$$

Assume that

$$S \in \mathcal{R}, \ \{x_k\}_{k=0}^\infty \subset X, \ \|x_0\| \leq M,$$

$$x_{k+1} = [S(k)](x_k), \ k = 0, 1, \dots.$$

Then

$$Card(\{i \in \{0, 1, \dots\} :\ x_i \notin \tilde{F}_\epsilon\}) \le 4M^2\bar{c}^{-1}l\epsilon^{-2}(\min\{l^{-1}, \bar{\lambda}\})^{-2}.$$

In Chapter 6 of this book we generalize this result for perturbed algorithms in the case when perturbations are summable. We also establish, for every positive number ϵ, the convergence of cyclic algorithms to the set of approximate fixed points $\tilde{F}_\epsilon$ in the presence of small computational errors.

1.4 Subgradient Projection Algorithms

In Chapter 6 we use subgradient projection algorithms for solving convex feasibility problems.

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, which induces a complete norm $\|\cdot\|$. For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) := \inf\{\|x - y\| :\ y \in A\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X :\ \|x - y\| \le r\}.$$

It is well known (see Fact 1.5 and Lemma 2.4 of [8]) that for each nonempty, closed, and convex subset C of X and for each $x \in X$, there is a unique point $P_C(x) \in C$ satisfying

$$\|x - P_C(x)\| = d(x, C).$$

Let $f : X \to R^1$ be a continuous and convex function such that

$$\{x \in X :\ f(x) \le 0\} \ne \emptyset.$$

Let $y_0 \in X$. Then the set

$$\partial f(y_0) := \{l \in X :\ f(y) - f(y_0) \ge \langle l, y - y_0\rangle \text{ for all } y \in X\}$$

is the subdifferential of f at the point y_0 [94, 99]. It is not difficult to see that for any $l \in \partial f(y_0)$,

$$\{x \in X :\ f(x) \le 0\} \subset \{x \in X :\ f(y_0) + \langle l, x - y_0\rangle \le 0\}.$$

It is well known that the following lemma holds (see Lemma 7.3 of [8]).

Lemma 1.7 *Let $y_0 \in X$, $f(y_0) > 0$, $l \in \partial f(y_0)$ and let*

$$D := \{x \in X : \ f(y_0) + \langle l, x - y_0 \rangle \le 0\}.$$

Then $l \neq 0$ and $P_D(y_0) = y_0 - f(y_0)\|l\|^{-2}l$.

Denote by $\mathcal{N}$ the set of all nonnegative integers. Let m be a natural number, $\mathbb{I} = \{1, \ldots, m\}$, and $f_i : X \to R^1$, $i \in \mathbb{I}$, be convex and continuous functions. For each $i \in \mathbb{I}$ set

$$C_i := \{x \in X : \ f_i(x) \le 0\},$$

$$C := \cap_{i \in \mathbb{I}} C_i = \cap_{i \in \mathbb{I}} \{x \in X : \ f_i(x) \le 0\}.$$

Suppose that

$$C \neq \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given $\epsilon > 0$, a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if $f_i(x) \le \epsilon$ for all $i \in \mathbb{I}$. We apply the subgradient projection method in order to obtain a good approximative solution of the feasibility problem.

Consider a natural number $\bar{p} \ge m$. Denote by $\mathbb{S}$ the set of all mappings $S : \mathcal{N} \to \mathbb{I}$ such that the following property holds:

(P1) For each integer $N \in \mathcal{N}$ and each $i \in \mathbb{I}$, there is $n \in \{N, \ldots, N + \bar{p} - 1\}$ such that $S(n) = i$.

We want to find approximate solutions of the inclusion $x \in C$. In order to meet this goal we apply algorithms generated by $S \in \mathbb{S}$.

For each $x \in X$, each number $\epsilon \ge 0$, and each $i \in \mathbb{I}$ set

$$A_i(x, \epsilon) := \{x\} \text{ if } f_i(x) \le \epsilon$$

and

$$A_i(x, \epsilon) := x - f_i(x)\{\|l\|^{-2}l : \ l \in \partial f_i(x)\} \text{ if } f_i(x) > \epsilon.$$

We associate with any $S \in \mathbb{S}$ the algorithm which generates, for any starting point $x_0 \in X$, a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ such that, for each integer $n \ge 0$,

$$x_{n+1} \in A_{S(n)}(x_n, 0).$$

It is not difficult to see that the sequence $\{x_n\}_{n=0}^{\infty}$ is well defined, and that for each integer $n \ge 0$, if $f_{S(n)}(x_n) > 0$, then $x_{n+1} = P_{D_n}(x_n)$, where

$$D_n = \{x \in X : \ f(x_n) + \langle l_n, x - x_n \rangle \le 0\} \text{ and } l_n \in \partial f_{S(n)}(x_n).$$

In Chapter 10 of [124] we proved the following result which shows that, for the subgradient projection method almost all iterates are good approximate solutions. Denote by Card(A) the cardinality of the set A.

Theorem 1.8 *Let*

$$b > 0,\ \epsilon \in (0, 1],\ \Lambda > 0,\ \gamma \in [0, \epsilon],$$

$$c \in B(0, b) \cap C,$$

$$|f_i(u) - f_i(v)| \le \Lambda \|u - v\|,\ u, v \in B(0, 3b + 1),\ i \in \mathbb{I},$$

let a positive number ϵ_0 *satisfy*

$$\epsilon_0 \le \epsilon \Lambda^{-1}$$

and let a natural number n_0 *satisfy*

$$4\bar{p}\epsilon_0^{-2} b^2 \le n_0.$$

Assume that

$$S \in \mathbb{S},\ x_0 \in B(0, b),$$

and that for each integer $n \ge 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \gamma).$$

Then

$$\|x_n\| \le 3b \text{ for all integers } n \ge 0$$

and

$$Card(\{N \in \mathcal{N} :\ \max\{\|x_{n+1} - x_n\| :\ n = N, \dots, N + \bar{p} - 1\} > \epsilon_0\}) \le n_0.$$

Moreover, if an integer $N \ge 0$ *satisfies*

$$\|x_{n+1} - x_n\| \le \epsilon_0,\ n = N, \dots, N + \bar{p} - 1,$$

then, for all integers $n, m \in \{N, \dots, N + \bar{p}\}$, $\|x_n - x_m\| \le \bar{p}\epsilon_0$ *and for all integers* $n = N, \dots, N + \bar{p}$ *and each* $i \in \mathbb{I}$, $f_i(x_n) \le \epsilon(\bar{p} + 1)$.

In Chapter 8 of this book we generalize this result for perturbed algorithms in the case when perturbations are summable.

1.5 Examples

In this section we consider examples of the problem discussed in Section 1.1 for which Theorems 1.1–1.4 can be applied. Let $(X, \|\cdot\|)$ be a Hilbert space equipped with the norm $\|\cdot\|$ which is induced by its inner product. Assume that P_i is the projection on a nonempty convex closed set $C_i \subset X$ for all $i = 1, \dots, 10$. Let us consider a mapping $r : \{1, 2, \dots\} \to \{1, \dots, 10\}$ such that $r(i) = i$, $i = 1, \dots, 10$ and $r(i + 10) = r(i)$ for all integers $i \geq 10$. We assume that

$$\emptyset \neq \cap_{i=1}^{10} C_i \subset \{z \in X : \ \|z\| \leq 10^4\}.$$

Example 1.9 Let $x_0 \in X$ satisfy $\|x_0\| \leq 10^4$ and let $\{x_i\}_{i=1}^{\infty} \subset X$ satisfy for each natural number i,

$$x_i = P_{r(i)}(x_{i-1}).$$

Let $\epsilon = 10^{-6}$. Applying Theorem 1.1 with $\bar{c} = 1$, $\bar{N} = 10$ and we obtain that

$$\mathrm{Card}(\{i \in \{0, 1, \dots\} : \ x_i \in \tilde{F}_\epsilon\}) \leq 4 \cdot 10^{23}.$$

This implies that there exists a nonnegative integer $j \leq 4 \cdot 10^{23}$ such that

$$x_j \in \tilde{F}_{10^{-6}}.$$

By the inclusion above, for all $i = 1, \dots, 10$

$$x_j \in \tilde{F}_\epsilon(P_i) \text{ and } d(x_j, F_\epsilon(P_i)) \leq 10^{-6}.$$

Let $i \in \{1, \dots, 10\}$. Then for each $y \in F_\epsilon(P_i))$,

$$\|x_j - P_i(x_j)\| \leq \|x_j - y\| + \|y - P_i(y)\| + \|P_i(y) - P_i(x_j)\|$$

$$\leq 2\|x_j - y\| + \epsilon$$

and

$$\|x_j - P_i(x_j)\| \leq 2d(x_j, F_\epsilon(P_i)) + \epsilon \leq 3 \cdot 10^{-6}.$$

Thus for all $i = 1, \dots, 10$,

$$\|x_j - P_i(x_j)\| \leq 3 \cdot 10^{-6}.$$

Example 1.10 Let $f : X \to R^1$ be a convex function such that

$$|f(z_1) - f(z_2)| \leq 10^4\|z_1 - z_2\| \text{ for all } z_1, z_2 \in X.$$

The relation above implies that

$$\partial f(z) \subset \{x \in X : \|z\| \le 10^4\} \text{ for all } z \in X.$$

Let $\beta_i = i^{-2}$, $i = 1, 2, \ldots$, $x_0 \in X$ satisfy $\|x_0\| \le 10^4$ and let for each natural number i, $x_i \in X$ satisfy

$$x_i = P_{r(i)}(x_{i-1} - \beta_i \xi_i), \text{ where } \xi_i \in \partial f(x_i).$$

It is easy to see that for each integer $i \ge 1$,

$$\|x_i - P_{r(i)}(x_{i-1})\| \le \beta_i \|\xi_i\| \le 10^4 \beta_i = 10^4 i^{-2}.$$

Let $\epsilon = 10^{-6}$. Theorem 1.2 can be applied with $\bar{N} = 10$, $\bar{c} = 1$, $\epsilon_i = 10^4 i^{-2}$, $i = 1, 2, \ldots$, and

$$\Lambda = 10^4 \sum_{i=1}^{\infty} i^{-2} < 2 \cdot 10^4.$$

We choose a natural number n_0 such that for each integer $i \ge n_0$,

$$\epsilon_i = 10^4 i^{-2} < (20)^{-1} 10^{-6}.$$

The relation above holds if and only if $n_0^{-2} < 20^{-1} \cdot 10^{-10}$. Fix $n_0 = 5 \cdot 10^5$. By Theorem 1.2,

$$\mathrm{Card}(\{i \in \{0, 1, \ldots\} : x_i \in \tilde{F}_\epsilon\})$$

$$\le 5 \cdot 10^5 + 4 \cdot 10^{15}((2 \cdot 10^4 + 2 \cdot 10^4)^2 + 2 \cdot 10^4 + (2 \cdot 10^4 + 2 \cdot 10^4)) < 10^{25}.$$

Therefore there exists a nonnegative integer $j \le 10^{25}$ such that $x_j \in \tilde{F}_{10^{-6}}$. As in Example 1.9, we can show that for all $i = 1, \ldots, 10$,

$$\|x_j - P_i(x_j)\| \le 3 \cdot 10^{-6}.$$

Example 1.11 Let $x_0 \in X$ satisfy $\|x_0\| \le 10^4$ and let for each natural number i, $x_i \in X$ satisfy

$$x_i = P_{r(i)}(x_{i-1}).$$

It is not difficult to see that all the assumptions of Theorem 1.3 hold and

$$x_i \in F_{10^{-6}}$$

for each integer

$$i \geq 10(64 \cdot 10^{42}(21)^2 + 1).$$

Example 1.12 Note that property (P1) in Theorem 1.3 holds with $M_* = 10^4$. Assume that $C_1 \subset \{x \in X : \|x\| \leq 10^4\}$. Then

$$F_1 \subset \{x \in X : \|x\| \leq 10^4 + 1\}.$$

Set

$$\tilde{M} = 10^4 + 1, \ \epsilon = 10^{-6},$$

$$q_0 = (10^4 + 1)^4 21^2 10^{26}$$

and

$$\delta = (80q_0)^{-1}\epsilon = (80q_0)^{-1}10^{-6}.$$

Let $x_0 \in X$ satisfy $\|x_0\| \leq 10^4$ and let for each natural number i, $x_i \in X$ satisfy

$$\|x_i - P_{r(i)}(x_{i-1})\| \leq \delta.$$

By Theorem 1.4, for all integers $i \geq 10q_0$,

$$x_i \in F_{10^{-6}}.$$

Chapter 2
Iterative Methods in Metric Spaces

In this chapter we study the convergence of iterative methods for solving common fixed point problems in a metric space. Our main goal is to obtain an approximate solution of the problem using perturbed algorithms. We show that the inexact iterative method generates an approximate solution if perturbations are summable. We also show that if the mappings are nonexpansive and the perturbations are sufficiently small, then the inexact method produces approximate solutions.

2.1 The First Problem

Let (X, d) be a metric space. For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{d(x, y) : \ y \in E\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ d(x, y) \le r\}.$$

Let m be a natural number, $\bar{c} \in (0, 1)$ and let $P_i : X \to X$, $i = 1, \dots, m$ be self-mappings of the space X. Suppose that for every $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \ne \emptyset. \tag{2.1}$$

We also suppose that for every $i \in \{1, \dots, m\}$ the inequality

$$d(z, x)^2 \ge d(z, P_i(x))^2 + \bar{c}d(x, P_i(x))^2 \tag{2.2}$$

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holds for all $x \in X$ and all $z \in \mathrm{Fix}(P_i)$. Set

$$F = \cap_{i=1}^{m} \mathrm{Fix}(P_i). \tag{2.3}$$

Elements of the set F are solutions of common fixed point problem. In this chapter we obtain its approximate solution. In order to meet this goal we introduce the following notation.

For every $\epsilon > 0$ and every $i \in \{1, \dots, m\}$ set

$$F_\epsilon(P_i) = \{x \in X : \ d(x, P_i(x)) \le \epsilon\}, \tag{2.4}$$

$$\tilde{F}_\epsilon(P_i) = \{y \in X : \ d(y, F_\epsilon(P_i)) \le \epsilon\}, \tag{2.5}$$

$$F_\epsilon = \cap_{i=1}^{m} F_\epsilon(P_i), \tag{2.6}$$

$$\tilde{F}_\epsilon = \cap_{i=1}^{m} \tilde{F}_\epsilon(P_i). \tag{2.7}$$

Fix $\theta \in X$ and a natural number $\bar{N} \ge m$.

Denote by $\mathcal{R}$ the set of all mappings $r : \{1, 2, \dots\} \to \{1, \dots, m\}$ such that for each natural number j,

$$\{1, \dots, m\} \subset \{r(j), \dots, r(j + \bar{N} - 1)\}. \tag{2.8}$$

Every $r \in \mathcal{R}$ generates the following algorithm.

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration point x_k calculate the next iteration point x_{k+1} by

$$x_{k+1} = P_{r(k+1)}(x_k).$$

Denote by $\mathrm{Card}(A)$ the cardinality of a set A. Suppose that the sum over empty set is zero.

In the next section we prove the following result obtained in [125], which shows that the inexact iterative method generates approximate solutions if perturbations are summable.

Theorem 2.1 *Assume that $M > 0$ satisfies*

$$B(\theta, M) \cap F \neq \emptyset, \tag{2.9}$$

ϵ is a positive number and that a sequence $\{\epsilon_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfies

$$\Lambda := \sum_{i=1}^{\infty} \epsilon_i < \infty. \tag{2.10}$$

Let a natural number n_0 be such that for all integers $i \geq n_0$,

$$\epsilon_i < (2\bar{N})^{-1}\epsilon. \tag{2.11}$$

Let

$$r \in \mathcal{R}, \tag{2.12}$$

$$x_0 \in B(\theta, M) \tag{2.13}$$

and let a sequence $\{x_i\}_{i=1}^{\infty} \subset X$ satisfy for each natural number i,

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \epsilon_i. \tag{2.14}$$

Then

$$Card(\{i \in \{0, 1, \dots\} : x_i \notin \tilde{F}_\epsilon\})$$

$$\leq n_0 + 4\bar{N}^3\bar{c}^{-1}\epsilon^{-2}((2M + \Lambda)^2 + 2\Lambda(2M + \Lambda)).$$

Example 2.2 Let (H, d) be a Hadamard space (see [5]), m be a natural number, and C_i, $i = 1, \dots, m$ be nonempty convex closed subsets of H. In view of Theorem 2.1.12 of [5], for every $x \in H$ and every $i \in \{1, \dots, m\}$, there exists a unique point $P_i(x) \in C_i$ such that

$$d(x, P_i(x)) = \inf\{d(x, z) : z \in C_i\}$$

and the mappings P_i, $i = 1, \dots, m$ satisfy (2.2). Therefore Theorem 2.1 can be applied in order to obtain an approximation of a common point of the sets C_i, $i = 1, \dots, m$.

2.2 Proof of Theorem 2.1

It follows from (2.9) that there exists

$$z \in B(\theta, M) \cap F. \tag{2.15}$$

Relations (2.13) and (2.15) imply that

$$d(x_0, z) \leq 2M. \tag{2.16}$$

Put

$$\epsilon_0 = 0. \tag{2.17}$$

We show that for all nonnegative integers i, we have

$$d(z, x_i) \le 2M + \sum_{j=0}^{i} \epsilon_j. \tag{2.18}$$

By (2.16) and (2.17), inequality (2.18) is true for $i = 0$.

Assume that $i \ge 0$ is an integer and that inequality (2.18) is valid. It follows from (2.2), (2.14), and (2.15) that

$$d(z, x_{i+1}) \le d(z, P_{r(i+1)}(x_i)) + d(P_{r(i+1)}(x_i), x_{i+1})$$

$$\le d(z, x_i) + \epsilon_{i+1} \le 2M + \sum_{j=0}^{i+1} \epsilon_j.$$

Therefore by induction we showed that inequality (2.18) is valid for all nonnegative integers i.

Put

$$\gamma_0 = \epsilon(2\bar{N})^{-1}. \tag{2.19}$$

By (2.11) and (2.19), for all integers $i \ge n_0$, we have

$$\epsilon_i < \gamma_0. \tag{2.20}$$

Let an integer $i \ge 0$ be given. By (2.2) and (2.15), we have

$$d(z, x_i)^2 - d(z, P_{r(i+1)}(x_i))^2 \ge \bar{c} d(x_i, P_{r(i+1)}(x_i))^2. \tag{2.21}$$

It follows from (2.2), (2.14), (2.15), and (2.18) that

$$|d(z, x_{i+1})^2 - d(z, P_{r(i+1)}(x_i))^2|$$

$$\le |d(z, x_{i+1}) - d(z, P_{r(i+1)}(x_i))| \times (d(z, x_{i+1}) + d(z, P_{r(i+1)}(x_i)))$$

$$\le d(x_{i+1}, P_{r(i+1)}(x_i))(d(z, x_{i+1}) + d(z, x_i))$$

$$\le 2\epsilon_{i+1}(2M + \sum_{j=0}^{\infty} \epsilon_j). \tag{2.22}$$

In view of (2.21) and (2.28),

$$\bar{c} d(x_i, P_{r(i+1)}(x_i))^2 \le d(z, x_i)^2 - d(z, P_{r(i+1)}(x_i))^2$$

$$\le d(z, x_i)^2 - d(z, x_{i+1})^2 + 2\epsilon_{i+1}(2M + \sum_{j=0}^{\infty} \epsilon_j). \tag{2.23}$$

It follows from (2.18) and (2.23) that for every natural number $n > n_0$, we have

$$(2M + \sum_{j=0}^{\infty} \epsilon_j)^2 \geq d(z, x_{n_0})^2$$

$$\geq d(z, x_{n_0})^2 - d(z, x_n)^2$$

$$= \sum_{k=n_0}^{n-1} [d(z, x_k)^2 - d(z, x_{k+1})^2]$$

$$\geq \sum_{k=n_0}^{n-1} [\bar{c} d(x_k, P_{r(k+1)}(x_k))^2 - 2\epsilon_{k+1}(2M + \sum_{j=0}^{\infty} \epsilon_j)]$$

and

$$(2M + \sum_{j=0}^{\infty} \epsilon_j)^2 + 2 \sum_{j=0}^{\infty} \epsilon_j (2M + \sum_{j=0}^{\infty} \epsilon_j)$$

$$\geq \sum_{k=n_0}^{n-1} \bar{c} d(x_k, P_{r(k+1)}(x_k))^2$$

$$\geq \bar{c} \gamma_0^2 \mathrm{Card}(\{k \in \{n_0, \dots, n-1\} : \ d(x_k, P_{r(k+1)}(x_k)) \geq \gamma_0\}).$$

Since the relation above holds for every natural number $n > n_0$ we conclude that

$$\mathrm{Card}(\{k \geq n_0 \text{ is an integer } : \ d(x_k, P_{r(k+1)}(x_k)) \geq \gamma_0\})$$

$$\leq \bar{c}^{-1} \gamma_0^{-2} ((2M + \sum_{j=0}^{\infty} \epsilon_j)^2 + 2 \sum_{j=0}^{\infty} \epsilon_j (2M + \sum_{j=0}^{\infty} \epsilon_j)). \tag{2.24}$$

Define

$$E_0 = \{k \in \{n_0, n_0 + 1, \dots\} : \ d(x_k, P_{r(k+1)}(x_k)) \geq \gamma_0\}. \tag{2.25}$$

By (2.10), (2.24), and (2.25), we have

$$\mathrm{Card}(E_0) \leq \bar{c}^{-1} \gamma_0^{-2} ((2M + \Lambda)^2 + 2\Lambda(2M + \Lambda)). \tag{2.26}$$

Define

$$E_1 = \{k \in \{n_0, n_0 + 1, \dots\} : [k, k + \bar{N} - 1] \cap E_0 \neq \emptyset\}. \tag{2.27}$$

It follows from (2.26) and (2.27) that

$$\begin{aligned} \mathrm{Card}(E_1) &\le \bar{N}\mathrm{Card}(E_0) \\ &\le \bar{N}\bar{c}^{-1}\gamma_0^{-2}((2M + \Lambda)^2 + 2\Lambda(2M + \Lambda)). \end{aligned} \tag{2.28}$$

Assume that a nonnegative integer p satisfies

$$p \ge n_0 \text{ and } p \notin E_1. \tag{2.29}$$

By (2.27) and (2.29),

$$[p, p + \bar{N} - 1] \cap E_0 = \emptyset$$

and for every integer $k \in \{p, \dots, p + \bar{N} - 1\}$, we have

$$d(x_k, P_{r(k+1)}(x_k)) < \gamma_0. \tag{2.30}$$

It follows from (2.14), (2.20), (2.29), and (2.30) that for every integer $k \in \{p, \dots, p + \bar{N} - 1\}$ we have

$$d(x_k, x_{k+1}) \le d(x_k, P_{r(k+1)}(x_k)) + d(P_{r(k+1)}(x_k), x_{k+1}) < \gamma_0 + \epsilon_{k+1} < 2\gamma_0. \tag{2.31}$$

By (2.31), for all pairs of integers $k_1, k_2 \in \{p, \dots, p + \bar{N}\}$,

$$d(x_{k_1}, x_{k_2}) \le 2\bar{N}\gamma_0. \tag{2.32}$$

Let $s \in \{1, \dots, m\}$ be given. In view of (2.8) and (2.12), there is an integer k which satisfies

$$k \in \{p, \dots, p + \bar{N} - 1\},\ r(k + 1) = s. \tag{2.33}$$

By (2.30) and (2.33), we have

$$d(x_k, P_s(x_k)) < \gamma_0. \tag{2.34}$$

Relations (2.32) and (2.33) imply that

$$d(x_p, x_k) \le 2\bar{N}\gamma_0.$$

In view of (2.19), (2.34) and the inequality above,

$$x_p \in \tilde{F}_{2\bar{N}\gamma_0}(P_s) = \tilde{F}_\epsilon(P_s),\ s = 1, \dots, m$$

and

$$x_p \in \tilde{F}_\epsilon$$

for each integer $p \geq n_0$ satisfying $p \neq E_1$. Thus

$$\mathrm{Card}(\{i \in \{0, 1, \dots\} :\ x_i \notin \tilde{F}_\epsilon\}) \leq n_0 + \mathrm{Card}(E_1)$$

$$\leq n_0 + 4\bar{N}^3\bar{c}^{-1}\epsilon^{-2}((2M + \Lambda)^2 + 2\Lambda(2M + \Lambda)).$$

This completes the proof of Theorem 2.1.

2.3 Cyclic Iterative Methods

In this section we use the notation and definitions introduced in Section 2.1. Assume that for every $i \in \{1, \dots, m\}$ and every pair of points $x, y \in X$,

$$d(P_i(x), P_i(y)) \leq d(x, y). \tag{2.35}$$

Suppose that $M_* > 1$ and that the following property holds:

(P1) for each $\delta > 0$ there exists $z_\delta \in B(\theta, M_*)$ such that

$$B(z_\delta, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m.$$

Denote by $\mathcal{R}_{per}$ the set of all $r \in \mathcal{R}$ such that for each integer $i \geq 1$,

$$r(i + \bar{N}) = r(i). \tag{2.36}$$

The following result shows that the exact iterative method generates approximate solutions.

Theorem 2.3 *Let $M \geq M_*,\ \epsilon > 0$,*

$$r \in \mathcal{R}_{per}, \tag{2.37}$$

$$x_0 \in B(\theta, M), \tag{2.38}$$

$\{x_i\}_{i=1}^\infty \subset X$ *and let for each natural number i,*

$$x_i = P_{r(i)}(x_{i-1}). \tag{2.39}$$

Then for every integer $i \geq \bar{N}(64M^4\bar{c}^{-3}\bar{N}^2\epsilon^{-4}(2\bar{N}+1)^2+1)$,

$$x_i \in F_\epsilon.$$

Proof Let

$$\gamma_0 = (\epsilon(2\bar{N}+1)^{-1})^2\bar{c}(12M\bar{N})^{-1}. \tag{2.40}$$

By (2.39), for each integer $i \geq 0$,

$$x_{i+1} = P_{r(i+1)}(x_i). \tag{2.41}$$

Set

$$P_r = P_{r(\bar{N})} \cdots P_{r(1)} = \prod_{i=1}^{\bar{N}} P_{r(i)}. \tag{2.42}$$

It follows from (2.36), (2.41), and (2.42) that for each integer $i \geq 0$,

$$x_{(i+1)\bar{N}} = P_{r((i+1)\bar{N})} \cdots P_{r(i\bar{N}+1)}(x_{i\bar{N}}) = P_r(x_{i\bar{N}}). \tag{2.43}$$

Let n be a natural number and δ be an arbitrary positive number. Property (P1) implies that there exists

$$z_\delta \in B(\theta, M_*) \tag{2.44}$$

such that

$$B(z_\delta, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset,\ \ i = 1, \ldots, m. \tag{2.45}$$

In view of (2.38) and (2.44),

$$d(z_\delta, x_0) \leq 2M. \tag{2.46}$$

In view of (2.45), for each integer $i \in \{1, \ldots, m\}$, there exists

$$z_{\delta,i} \in \mathrm{Fix}(P_i) \tag{2.47}$$

such that

$$d(z_\delta, z_{\delta,i}) \leq \delta. \tag{2.48}$$

By (2.35), (2.47), and (2.48), for each integer $i \in \{1, \dots, m\}$,

$$d(z_\delta, P_i(z_\delta)) \le d(z_\delta, z_{\delta,i}) + d(P_i(z_{\delta,i}), P_i(z_\delta)) \le 2d(z_\delta, z_{\delta,i}) \le 2\delta. \tag{2.49}$$

It follows from (2.35), (2.41), and (2.49) that for each integer $k \ge 0$,

$$d(z_\delta, x_{k+1}) = d(z_\delta, P_{r(k+1)}(x_k))$$

$$\le d(z_\delta, P_{r(k+1)}(z_\delta)) + d(P_{r(k+1)}(z_\delta), P_{r(k+1)}(x_k)) \le 2\delta + d(z_\delta, x_k). \tag{2.50}$$

Relations (2.46) and (2.50) imply that for all integers $k = 0, \dots, n$,

$$d(z_\delta, x_k) \le d(z_\delta, x_0) + 2\delta k \le 2M + \delta n. \tag{2.51}$$

In view of (2.44) and (2.51), for all integers $k = 0, \dots, n$,

$$d(\theta, x_k) \le d(\theta, z_\delta) + d(z_\delta, x_k) \le 3M + \delta n.$$

Since δ is an arbitrary positive number we conclude that

$$d(\theta, x_k) \le 3M \text{ for all integers } k \ge 0. \tag{2.52}$$

Let n be a natural number and a positive number ϵ_0 satisfies

$$\epsilon_0 < (2n\bar{N})^{-1}. \tag{2.53}$$

Since the function

$$(\xi_1, \xi_2) \to d(\xi_1, \xi_2)^2, \ (\xi_1, \xi_2) \in X \times X$$

is uniformly continuous on bounded subsets of $X \times X$, there exist $\delta \in (0, 1)$ such that for each

$$(\xi_1, \xi_2), \ (\eta_1, \eta_2) \in B(\theta, 3M + 1) \times B(\theta, 3M + 1)$$

satisfying

$$d(\xi_i, \eta_i) \le \delta, \ i = 1, 2$$

the following inequality holds:

$$|d(\xi_1, \xi_2)^2 - d(\eta_1, \eta_2)^2| \le \epsilon_0. \tag{2.54}$$

Property (P1) and (2.55) imply that there exists

$$z \in B(\theta, M) \tag{2.55}$$

such that

$$B(z, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m. \tag{2.56}$$

In view of (2.56), for every $i \in \{1, \dots, m\}$, there exists

$$z_i \in \mathrm{Fix}(P_i) \tag{2.57}$$

such that

$$d(z, z_i) \leq \delta. \tag{2.58}$$

Let $k \geq 0$ be an integer. By (2.52), (2.55), (2.58), and the choice of δ (see (2.54)),

$$|d(z, x_k)^2 - d(z_{r(k+1)}, x_k)^2| \leq \epsilon_0, \tag{2.59}$$

$$|d(z, x_{k+1})^2 - d(z_{r(k+1)}, x_{k+1})^2| \leq \epsilon_0. \tag{2.60}$$

It follows from (2.2), (2.41), (2.57), (2.59), and (2.60) that

$$\begin{aligned} & d(z, x_k)^2 - d(z, x_{k+1})^2 \\ & \geq d(z_{r(k+1)}, x_k)^2 - d(z_{r(k+1)}, x_{k+1})^2 - 2\epsilon_0 \\ & = d(z_{r(k+1)}, x_k)^2 - d(z_{r(k+1)}, P_{r(k+1)}(x_k))^2 - 2\epsilon_0 \\ & \geq \bar{c} d(x_k, x_{k+1})^2 - 2\epsilon_0 \end{aligned} \tag{2.61}$$

for all integers $k \geq 0$.

By (2.52), (2.55), and (2.61),

$$\begin{aligned} 16M^2 & \geq d(z, x_0)^2 \geq d(z, x_0)^2 - d(z, x_{n\bar{N}})^2 \\ & = \sum_{k=0}^{n-1} [d(z, x_{k\bar{N}})^2 - d(z, x_{(k+1)\bar{N}})^2] \\ & = \sum_{k=0}^{n-1} [\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (d(z, x_j)^2 - d(z, x_{j+1})^2)] \end{aligned}$$

$$\geq \sum_{k=0}^{n-1}[\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1}(\bar{c}d(x_j,x_{j+1})^2-2\epsilon_0)]$$

$$=\sum_{k=0}^{n-1}[\bar{c}\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1}d(x_j,x_{j+1})^2]-2\epsilon_0 n\bar{N}$$

$$\geq \bar{c}\gamma_0^2\mathrm{Card}(\{k\in\{0,\dots,n-1\}:$$

$$\max\{d(x_j,x_{j+1}):\ j=k\bar{N},\dots,(k+1)\bar{N}-1\}\geq\gamma_0\})-2\epsilon_0 n\bar{N}.$$

Together with (2.53) this implies that

$$\mathrm{Card}(\{k\in\{0,\dots,n-1\}:$$

$$\max\{d(x_j,x_{j+1}):\ j=k\bar{N},\dots,(k+1)\bar{N}-1\}\geq\gamma_0\})\leq(16M^2+1)\bar{c}^{-1}\gamma_0^{-2}.$$

Since the inequality above holds for any natural number n we conclude that

$$\mathrm{Card}(\{k\in\{0,1,\dots,\}:$$

$$\max\{d(x_j,x_{j+1}):\ j=k\bar{N},\dots,(k+1)\bar{N}-1\}\geq\gamma_0\})\leq(16M^2+1)\bar{c}^{-1}\gamma_0^{-2}. \tag{2.62}$$

By (2.62), there exists an integer $q_0\geq 0$ such that

$$q_0\leq(16M^2+1)\bar{c}^{-1}\gamma_0^{-2}+1 \tag{2.63}$$

and

$$d(x_j,x_{j+1})\leq\gamma_0,\ \ j=q_0\bar{N},\dots,(q_0+1)\bar{N}-1. \tag{2.64}$$

In view of (2.43) and (2.64),

$$d(x_{q_0\bar{N}},P_r(x_{q_0\bar{N}}))=d(x_{q_0\bar{N}},x_{(q_0+1)\bar{N}})\leq\gamma_0\bar{N}. \tag{2.65}$$

It follows from (2.35), (2.42), and (2.65) that for each integer $q>q_0$,

$$d(x_{q\bar{N}},x_{(q+1)\bar{N}})=d((P_r)^{q-q_0}(x_{q_0\bar{N}}),(P_r)^{q-q_0}(x_{(q_0+1)\bar{N}}))$$

$$\leq d(x_{q_0\bar{N}},x_{(q_0+1)\bar{N}})\leq\gamma_0\bar{N}. \tag{2.66}$$

Let $q \geq q_0$ be an integer. By (2.65) and (2.66),

$$d(x_{q\bar{N}}, x_{(q+1)\bar{N}}) \leq \gamma_0 \bar{N}. \tag{2.67}$$

Choose a positive number

$$\delta < \min\{\gamma_0/4, 1\}. \tag{2.68}$$

Property (P1) implies that there exists

$$z \in B(\theta, M) \tag{2.69}$$

such that

$$B(z, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset, \ i = 1, \ldots, m. \tag{2.70}$$

In view of (2.70), for each $i \in \{1, \ldots, m\}$, there exists

$$z_i \in \mathrm{Fix}(P_i) \tag{2.71}$$

such that

$$d(z, z_i) \leq \delta. \tag{2.72}$$

It follows from (2.52), (2.67), and (2.69) that

$$\begin{aligned} \gamma_0 \bar{N} &\geq d(x_{q\bar{N}}, x_{(q+1)\bar{N}}) \\ &\geq d(z, x_{q\bar{N}}) - d(z, x_{(q+1)\bar{N}}) \\ &\geq (8M)^{-1}(d(z, x_{q\bar{N}})^2 - d(z, x_{(q+1)\bar{N}})^2) \end{aligned}$$

and

$$\begin{aligned} 8M\gamma_0 \bar{N} &\geq d(z, x_{q\bar{N}})^2 - d(z, x_{(q+1)\bar{N}})^2 \\ &= \sum_{i=q\bar{N}}^{(q+1)\bar{N}-1} (d(z, x_i)^2 - d(z, x_{+1})^2). \end{aligned} \tag{2.73}$$

Let

$$i \in \{q\bar{N}, \ldots, (q+1)\bar{N} - 1\}. \tag{2.74}$$

By (2.52), (2.69), and (2.72),

$$|d(z, x_i)^2 - d(z_{r(i+1)}, x_i)^2|$$
$$\leq (d(z, x_i) + d(z_{r(i+1)}, x_i))d(z, z_{r(i+1)}) \leq 9\delta M, \tag{2.75}$$
$$|d(z, x_{i+1})^2 - d(z_{r(i+1)}, x_{i+1})^2|$$
$$\leq (d(z, x_{i+1}) + d(z_{r(i+1)}, x_{i+1}))d(z, z_{r(i+1)}) \leq 9\delta M. \tag{2.76}$$

It follows from (2.2), (2.41), (2.71), (2.75), and (2.76) that

$$d(z, x_i)^2 - d(z, x_{i+1})^2$$
$$\geq d(z_{r(i+1)}, x_i)^2 - d(z_{r(i+1)}, P_{r(i+1)}(x_i))^2 - 18\delta M \geq \bar{c}d(x_i, x_{i+1})^2 - 18\delta M. \tag{2.77}$$

Relations (2.73), (2.74), and (2.77) imply that

$$8M\gamma_0\bar{N} \geq \sum_{i=q\bar{N}}^{(q+1)\bar{N}-1} (d(z, x_i)^2 - d(z, x_{+1})^2)$$
$$\geq \bar{c} \sum_{i=q\bar{N}}^{(q+1)\bar{N}-1} d(x_i, x_{i+1})^2 - 18\delta M\bar{N}. \tag{2.78}$$

Since δ is any positive number satisfying (2.68) it follows from (2.78) that

$$8M\gamma_0\bar{N}\bar{c}^{-1} \geq \sum_{i=q\bar{N}}^{(q+1)\bar{N}-1} d(x_i, x_{i+1})^2$$

and for each $i = q\bar{N}, \ldots, (q+1)\bar{N} - 1$,

$$d(x_i, x_{i+1}) \leq (8M\gamma_0\bar{N}\bar{c}^{-1})^{1/2}. \tag{2.79}$$

In view of (2.79), for each $i, j \in \{q\bar{N}, \ldots, (q+1)\bar{N}\}$,

$$d(x_i, x_j) \leq \bar{N}(8M\gamma_0\bar{N}\bar{c}^{-1})^{1/2}. \tag{2.80}$$

Let $s \in \{1, \ldots, m\}$. By (2.8) and (2.37), there exists

$$j \in \{q\bar{N}, \ldots, (q+1)\bar{N} - 1\} \tag{2.81}$$

such that

$$r(j+1) = s. \tag{2.82}$$

By (2.41), (2.79), (2.81), and (2.82),

$$d(x_j, P_s(x_j)) = d(x_j, P_{r(j+1)}(x_j))$$
$$= d(x_j, x_{j+1}) \le (8M\gamma_0\bar{N}\bar{c}^{-1})^{1/2}. \tag{2.83}$$

It follows from (2.35), (2.40), (2.80), and (2.83) that for each $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$d(x_i, P_s(x_i)) \le d(x_i, x_j) + d(x_j, P_s(x_j)) + d(P_s(x_j), P_s(x_i))$$
$$\le 2d(x_i, x_j) + (8M\gamma_0\bar{N}\bar{c}^{-1})^{1/2} \le (8M\gamma_0\bar{N}\bar{c}^{-1})^{1/2}(2\bar{N}+1) \le \epsilon$$

and since s is any integer belonging to $\{1, \dots, m\}$ we conclude in view of (2.4) and (2.6), that

$$x_i \in F_\epsilon$$

for all integers $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$. Since q is any integer satisfying $q \ge q_0$ this completes the proof of Theorem 2.3.

2.4 Cyclic Iterative Methods with Computational Errors

In this section we use the notation and assumptions introduced in Section 2.3. For a real number $z \in R^1$ we set $\lfloor z \rfloor = \max\{i : \ i \text{ is an integer and } i \le z\}$.

The following result shows that the inexact iterative method generates approximate solutions if the perturbations are small enough.

Theorem 2.4 *Suppose that $\tilde{M} > M_*$, $r_0 > 0$ and*

$$F_{r_0} \subset B(\theta, \tilde{M}). \tag{2.84}$$

Let $M \ge \tilde{M}$, $\epsilon \in (0, r_0]$,

$$q_0 = 3 + \lfloor 4^7 M^4 \bar{N}^2 (2\bar{N}+1)^4 \bar{c}^{-3} \epsilon^{-4} \rfloor, \tag{2.85}$$

a positive number δ satisfy

$$2\delta q_0 \bar{N} \le \epsilon/4, \tag{2.86}$$

$$r \in \mathcal{R}_{per}, \tag{2.87}$$

$$x_0 \in B(\theta, M)$$

and let $\{x_i\}_{i=1}^{\infty} \subset X$ *satisfy for each natural number* i,

$$d(x_i, P_{r(i)}(x_{i-1})) \le \delta. \tag{2.88}$$

Then for every integer $i \ge q_0 \bar{N}$,

$$x_i \in F_\epsilon.$$

Proof Assume that $n \ge 0$ is an integer and that

$$x_{n\bar{N}} \in B(\theta, M). \tag{2.89}$$

Consider a sequence $\{y_i\}_{i=n\bar{N}}^{\infty} \subset X$ such that

$$y_{n\bar{N}} = x_{n\bar{N}} \tag{2.90}$$

and that for each integer $i \ge n\bar{N} + 1$,

$$y_i = P_{r(i)}(y_{i-1}). \tag{2.91}$$

By (2.87), (2.89)–(2.91), and Theorem 2.3, for each integer

$$i \ge \bar{N}(4^7 M^4 \bar{N}^2 (2\bar{N} + 1)^4 \bar{c}^{-3} \epsilon^{-4} + 1) + n\bar{N}$$

we have

$$y_i \in F_{\epsilon/4}. \tag{2.92}$$

In view of (2.85) and (2.92),

$$y_i \in F_{\epsilon/4} \text{ for each } i \in \{n\bar{N} + q_0\bar{N}, \ldots, n\bar{N} + 2q_0\bar{N}\}. \tag{2.93}$$

We show by induction that for each integer $i \ge n\bar{N}$,

$$d(x_i, y_i) \le \delta(i - \bar{N}n). \tag{2.94}$$

By (2.90), inequality (2.94) is true for $i = n\bar{N}$. Assume that $i \ge \bar{N}n$ is an integer and that (2.94) is true. It follows from (2.35), (2.88), (2.91), and (2.94) that

$$d(x_{i+1}, y_{i+1}) \le d(x_{i+1}, P_{r(i+1)}(x_i)) + d(P_{r(i+1)}(x_i), P_{r(i+1)}(y_i))$$

$$\le \delta + d(x_i, y_i) \le \delta(i + 1 - \bar{N}n).$$

Thus (2.94) holds for all integers $i \ge n\bar{N}$.

By (2.86) and (2.94), for all integers $i \in \{n\bar{N} + q_0\bar{N}, \dots, n\bar{N} + 2q_0\bar{N}\}$

$$d(x_i, y_i) \le 2\delta q_0 \bar{N} \le \epsilon/4. \tag{2.95}$$

Let

$$i \in \{n\bar{N} + q_0\bar{N}, \dots, n\bar{N} + 2q_0\bar{N}\}.$$

By (2.93),

$$y_i \in F_{\epsilon/4}. \tag{2.96}$$

In view of (2.15),

$$d(x_i, y_i) \le \epsilon/4. \tag{2.97}$$

Relations (2.4), (2.6), (2.35), (2.95), and (2.96) imply that for every $s \in \{1, \dots, m\}$,

$$d(x_i, P_s(x_i)) \le d(x_i, y_i) + d(y_i, P_s(y_i)) + d(P_s(y_i), P_s(x_i))$$

$$\le \epsilon/4 + 2d(x_i, y_i) \le 3\epsilon/4,$$

$$x_i \in F_\epsilon(P_s)$$

and

$$x_i \in F_\epsilon.$$

Thus we have shown that the following property holds:

(P2) if $n \ge 0$ is an integer and

$$x_{n\bar{N}} \in B(\theta, M),$$

then

$$x_i \in F_\epsilon,\ i = n\bar{N} + q_0\bar{N}, \dots, n\bar{N} + 2q_0\bar{N}.$$

Property (P2) and (2.36) imply that

$$x_i \in F_\epsilon,\ i = q_0\bar{N}, \ldots, 2q_0\bar{N}. \tag{2.98}$$

Assume that an integer $q \geq q_0$ and that

$$x_i \in F_\epsilon,\ i = q_0\bar{N}, \ldots, q\bar{N} + q_0\bar{N}. \tag{2.99}$$

(Note that in view of (2.98) our assumption holds for $q = q_0$.) By (2.85) and (2.90),

$$x_{q\bar{N}} \in F_\epsilon \subset F_{r_0} \subset B(\theta, \tilde{M}) \subset B(\theta, M). \tag{2.100}$$

Property (P2) and (2.100) imply that

$$x_i \in F_\epsilon,\ i = q\bar{N} + q_0\bar{N}, \ldots, q\bar{N} + 2q_0\bar{N}.$$

Together with (2.99) this implies that

$$x_i \in F_\epsilon,\ i = q_0\bar{N}, \ldots, (q + 2q_0)\bar{N}.$$

This implies that $x_i \in F_\epsilon$ for all integers $i \geq q_0\bar{N}$. Theorem 2.4 is proved.

2.5 The Second Problem

Let (X, d) be a metric space. Recall that for each $x \in X$ and each $r > 0$,

$$B(x, r) = \{y \in X :\ d(x, y) \leq r\}$$

and for each $x \in X$ and each nonempty set $E \subset X$,

$$d(x, E) = \inf\{d(x, y) :\ y \in E\}.$$

Fix $\theta \in X$. Let m be a natural number, $C_i \subset X$, $i = 1, \ldots, m$ be nonempty closed sets and let $P_i : X \to X$, $i = 1, \ldots, m$ be self-mappings of the space X such that for every $i \in \{1, \ldots, m\}$,

$$P_i(x) = x \text{ for all } x \in C_i. \tag{2.101}$$

Suppose that the following assumption holds:

(A1) For each $M > 0$ and each $\gamma > 0$ there exists $\delta > 0$ such that for each $i \in \{1, \ldots, m\}$, each $x \in B(\theta, M)$ satisfying $d(x, C_i) \geq \gamma$, and each

$$z \in B(\theta, M) \cap C_i$$

the inequality

$$d(P_i(x), z) \le d(x, z) - \delta$$

is true.

In view of (A1) and (2.101), for each $i = 1, \dots, m$,

$$\{x \in X : \; P_i(x) = x\} = C_i, \tag{2.102}$$

$$d(P_i(x), z) \le d(x, z) \text{ for each } x \in X \text{ and each } z \in C_i. \tag{2.103}$$

Fix a natural number $\bar{N} \ge m$. Denote by $\mathcal{R}$ the set of all mappings $r : \{1, 2, \dots\} \to \{1, \dots, m\}$ such that for each number j,

$$\{1, \dots, m\} \subset \{r(j), \dots, r(j + \bar{N} - 1)\} \tag{2.104}$$

and denote by $\mathcal{R}_{per}$ the set of all $r \in \mathcal{R}$ such that for each integer $i \ge 1$,

$$r(i + \bar{N}) = r(i). \tag{2.105}$$

Suppose that $M_* > 1$ and that the following assumption holds:

(A2) for each $\delta > 0$ there exists $z_\delta \in B(\theta, M_*)$ such that

$$B(z_\delta, \delta) \cap C_i \ne \emptyset \text{ for all } i = 1, \dots, m.$$

In this chapter we prove the following three results: Theorem 2.5 which shows that the inexact iterative method generates approximate solutions if perturbations are summable, Theorem 2.6 which establishes that the exact iterative method generates approximate solutions, and Theorem 2.7 which demonstrates that the inexact iterative method generates approximate solutions if the perturbations are small enough.

Theorem 2.5 *Let $M \ge M_*$, ϵ be a positive number and let a sequence $\{\epsilon_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy*

$$\Lambda := \sum_{i=1}^{\infty} \epsilon_i < \infty. \tag{2.106}$$

Then there exists a constant $Q > 0$ such that for each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies

$$x_0 \in B(\theta, M),$$

$$d(x_i, P_{r(i)}(x_{i-1})) \le \epsilon_i \textit{ for all natural numbers } i$$

the inequality

$$Card(\{i \in \{0, 1, \dots\} : \max\{d(x_i, C_s) : s = 1, \dots, m\} > \epsilon\}) \leq Q$$

holds.

Theorem 2.6 *Assume that the following property holds:*

(A3) $d(P_i(x), P_i(y)) \leq d(x, y)$ *for all* $x, y \in X$ *and all* $i = 1, \dots, m$.

Let $M \geq M_*$, $\epsilon > 0$. *Then there exists a constant* $Q > 0$ *such that for each* $r \in \mathcal{R}_{per}$ *and each sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *which satisfies*

$$x_0 \in B(\theta, M),$$

$$x_i = P_{r(i)}(x_{i-1}) \text{ for all natural numbers } i$$

the inequality

$$\max\{d(x_i, C_s) : s = 1, \dots, m\} \leq \epsilon$$

holds for all integers $i \geq Q$.

Theorem 2.7 *Assume that (A3) holds. Let* $M \geq M_*$, $r_0 > 0$,

$$\{x \in X : d(x, C_s) \leq r_0, \ s = 1, \dots, m\} \subset B(\theta, M)$$

and $\epsilon_0 > 0$. *Then there exist* $Q, \delta > 0$ *such that for each* $r \in \mathcal{R}_{per}$ *and each sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *which satisfies*

$$x_0 \in B(\theta, M),$$

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \delta \text{ for all natural numbers } i$$

the inequality

$$\max\{d(x_i, C_s) : s = 1, \dots, m\} \leq \epsilon_0$$

holds for all integers $i \geq Q$.

2.6 Proof of Theorem 2.5

Set

$$\gamma_0 = \epsilon(2\bar{N} + 1)^{-1}. \tag{2.107}$$

By (A1), there exists a positive number $\gamma < \min\{1, \gamma_0\}$ such that the following property holds:

(P3) for each $i \in \{1, \dots, m\}$, each $z \in B(\theta, 3M + 1 + \Lambda) \cap C_i$, and each $x \in B(\theta, 3M + 1 + \Lambda)$ satisfying $d(x, C_i) \geq \gamma_0/2$ we have

$$d(P_i(x), z) \leq d(x, z) - \gamma.$$

By (2.106), there exists a natural number n_0 such that

$$\epsilon_i < \gamma/4 \text{ for all integers } i \geq n_0. \tag{2.108}$$

Set

$$Q = n_0 + 2\bar{N}\gamma^{-1}(4M + 2\Lambda). \tag{2.109}$$

Assume that $r \in \mathcal{R}$, $\{x_i\}_{i=0}^{\infty} \subset X$,

$$x_0 \in B(\theta, M) \tag{2.110}$$

and

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \epsilon_i \text{ for all natural numbers } i. \tag{2.111}$$

Set

$$\epsilon_0 = 0. \tag{2.112}$$

Let n be a natural number and $\delta > 0$. By (A2), there exists

$$z_\delta \in B(\theta, M_*) \tag{2.113}$$

such that

$$B(z_\delta, \delta) \cap C_i \neq \emptyset \text{ for all } i = 1, \dots, m. \tag{2.114}$$

By (2.114), for each $i \in \{1, \dots, m\}$ there exists

$$z_{\delta,i} \in C_i \tag{2.115}$$

such that

$$d(z_\delta, z_{\delta,i}) \leq \delta. \tag{2.116}$$

It follows from (2.103), (2.111), (2.115), and (2.116) that for each integer $i \geq 0$,

$$\begin{aligned} d(z_\delta, x_{i+1}) &\leq d(z_\delta, z_{\delta,r(i+1)}) + d(z_{\delta,r(i+1)}, x_{i+1}) \\ &\leq \delta + d(z_{\delta,r(i+1)}, P_{r(i+1)}(x_i)) + d(P_{r(i+1)}(x_i), x_{i+1}) \\ &\leq \delta + \epsilon_{i+1} + d(z_{\delta,r(i+1)}, x_i) \\ &\leq \delta + \epsilon_{i+1} + d(x_i, z_\delta) + d(z_\delta, z_{\delta,r(i+1)}) \\ &\leq 2\delta + \epsilon_{i+1} + d(z_\delta, x_i). \end{aligned} \tag{2.117}$$

By induction we show that for all integers $k = 0, \ldots, n$,

$$d(z_\delta, x_k) \leq 2M + \sum_{i=0}^{k} \epsilon_i + 2\delta k. \tag{2.118}$$

In view of (2.110), (2.112), and (2.113), the inequality (2.118) holds for $k = 0$. Assume that a nonnegative integer $k < n$ and that (2.118) holds. It follows from (2.117) and (2.118) that

$$\begin{aligned} d(z_\delta, x_{k+1}) &\leq 2\delta + \epsilon_{k+1} + d(z_\delta, x_k) \\ &\leq 2M + \sum_{i=0}^{k+1} \epsilon_i + 2\delta(k+1). \end{aligned}$$

Thus (2.118) holds for all $k = 0, \ldots, n$. It follows from (2.113) and (2.118) that for all $k = 0, \ldots, n$,

$$d(\theta, x_k) \leq d(\theta, z_\delta) + d(z_\delta, x_k) \leq 3M + \sum_{i=0}^{n} \epsilon_i + 2\delta n.$$

Since δ is any positive number we conclude (see (2.106)) that

$$d(\theta, x_k) \leq 3M + \Lambda \text{ for all integers } k \geq 0. \tag{2.119}$$

Set

$$E_0 = \{i \in \{n_0, n_0 + 1, \ldots\} : \ d(x_i, C_{r(i+1)}) \geq \gamma_0/2\}, \tag{2.120}$$

$$E_1 = \{n_0, n_0 + 1, \ldots\} \setminus E_0. \tag{2.121}$$

Let $n > n_0$ be an integer $\delta \in (0, 1)$. By (A2), there exists

$$z_\delta \in B(\theta, M_*) \tag{2.122}$$

such that (2.114) holds. By (2.114), for each $i \in \{1, \dots, m\}$ there exists

$$z_{\delta,i} \in C_i \tag{2.123}$$

such that

$$d(z_\delta, z_{\delta,i}) \le \delta. \tag{2.124}$$

Clearly, (2.117) holds for each integer $i \ge 0$. Therefore

$$d(z_\delta, x_{i+1}) \le d(z_\delta, x_i) + 2\delta + \epsilon_{i+1} \text{ for all integers } i \ge 0. \tag{2.125}$$

Let

$$i \in E_0. \tag{2.126}$$

Property (P3), (2.119), (2.120), (2.122)–(2.124), and (2.126) imply that

$$d(P_{r(i+1)}(x_i), z_{\delta,r(i+1)}) \le d(x_i, z_{\delta,r(i+1)}) - \gamma. \tag{2.127}$$

It follows from (2.124) and (2.127) that

$$\begin{aligned} d(P_{r(i+1)}(x_i), z_\delta) &\le d(P_{r(i+1)}(x_i), z_{\delta,r(i+1)}) + d(z_{\delta,r(i+1)}, z_\delta) \\ &\le \delta + d(x_i, z_{\delta,r(i+1)}) - \gamma \\ &\le d(x_i, z_\delta) + 2\delta - \gamma. \end{aligned} \tag{2.128}$$

By (2.108), (2.111), (2.120), (2.126), and (2.128),

$$\begin{aligned} d(x_{i+1}, z_\delta) &\le d(x_{i+1}, P_{r(i+1)}(x_i)) + d(P_{r(i+1)}(x_i), z_\delta) \\ &\le d(x_i, z_\delta) + 2\delta - \gamma + \epsilon_{i+1} \\ &\le d(x_i, z_\delta) + 2\delta - \gamma/2. \end{aligned} \tag{2.129}$$

By (2.119)–(2.122), (2.125), and (2.129),

$$\begin{aligned} 4M + \Lambda \ge d(z_\delta, x_{n_0}) &\ge d(z_\delta, x_{n_0}) - d(z_\delta, x_n) \\ &= \sum_{i=n_0}^{n-1} (d(z_\delta, x_i) - d(z_\delta, x_{i+1})) \\ &= \sum \{d(z_\delta, x_i) - d(z_\delta, x_{i+1}) :\ i \in E_1 \cap [0, n-1]\} \end{aligned}$$

$$+\sum\{d(z_\delta, x_i) - d(z_\delta, x_{i+1}) : i \in E_0 \cap [0, n-1]\}$$
$$\geq \sum\{-2\delta - \epsilon_{i+1} : i \in E_1 \cap [0, n-1]\}$$
$$+(\gamma/2 - 2\delta)\mathrm{Card}(E_0 \cap [0, n-1]). \quad (2.130)$$

In view of (2.106) and (2.130),

$$(\gamma/2 - 2\delta)\mathrm{Card}(E_0 \cap [0, n-1]) \leq 4M + 2\Lambda + 2\delta n.$$

Since δ is any element of the interval (0, 1) we conclude that

$$\mathrm{Card}(E_0 \cap [0, n-1]) \leq 2\gamma^{-1}(4M + 2\Lambda).$$

Since n is any integer satisfying $n > n_0$ we conclude that

$$\mathrm{Card}(E_0) \leq 2\gamma^{-1}(4M + 2\Lambda). \quad (2.131)$$

Set

$$E_2 = \{k \in \{n_0, n_0 + 1, \dots\} : [k, k + \bar{N} - 1] \cap E_0 \neq \emptyset\}. \quad (2.132)$$

By (2.131) and (2.132),

$$\mathrm{Card}(E_2) \leq \bar{N}\mathrm{Card}(E_0)$$
$$\leq 2\bar{N}\gamma^{-1}(4M + 2\Lambda)). \quad (2.133)$$

Let a nonnegative integer p satisfies

$$p \geq n_0 \text{ and } p \notin E_2. \quad (2.134)$$

Then in view of (2.120), (2.132), and (2.134),

$$[p, p + \bar{N} - 1] \cap E_0 = \emptyset \quad (2.135)$$

and for each $k \in \{p, \dots, p + \bar{N} - 1\}$,

$$d(x_k, C_{r(k+1)}) < \gamma_0/2. \quad (2.136)$$

Let

$$k \in \{p, \dots, p + \bar{N} - 1\}. \quad (2.137)$$

By (2.136) and (2.137), there exists

$$\xi \in C_{r(k+1)} \tag{2.138}$$

such that

$$d(x_k, \xi) < \gamma_0/2. \tag{2.139}$$

In view of (2.103), (2.111), (2.138), and (2.139),

$$d(x_{k+1}, \xi) \leq d(x_{k+1}, P_{r(k+1)}(x_k)) + d(P_{r(k+1)}(x_k), \xi)$$
$$\leq \epsilon_{k+1} + d(x_k, \xi) \leq \epsilon_{k+1} + \gamma_0/2.$$

Together with (2.108), (2.134), (2.137), and (2.139) this implies that

$$d(x_k, x_{k+1}) \leq d(x_k, \xi) + d(\xi, x_{k+1}) < \gamma_0 + \epsilon_{k+1} < 2\gamma_0. \tag{2.140}$$

In view of (2.137) and (2.140), for all integers $k_1, k_2 \in \{p, \dots, p + \bar{N}\}$,

$$d(x_{k_1}, x_{k_2}) \leq 2\bar{N}\gamma_0. \tag{2.141}$$

Let $s \in \{1, \dots, m\}$. By (2.104) and (2.110), there exists

$$k \in \{p, \dots, p + \bar{N} - 1\} \tag{2.142}$$

such that

$$r(k+1) = s. \tag{2.143}$$

It follows from (2.136), (2.142), and (2.143) that

$$d(x_k, C_s) < \gamma_0/2. \tag{2.144}$$

By (2.107), (2.141), (2.142), and (2.144),

$$d(x_p, x_k) \leq 2\bar{N}\gamma_0$$

and

$$d(x_p, C_s) \leq d(x_p, x_k) + d(x_k, C_s) \leq \gamma_0(2\bar{N} + 1) = \epsilon$$

for all $s \in \{1, \dots, m\}$. Thus for every integer $p \geq n_0$ such that $p \neq E_2$,

$$d(x_p, C_s) \leq \epsilon, \ \ s = 1, \dots, m.$$

Therefore in view of (2.109) and (2.133),

$$\begin{aligned}
&\mathrm{Card}(\{i \in \{0, 1, \dots\} : \max\{d(x_i, C_s) : s = 1, \dots, m\} > \epsilon\}) \\
&\le n_0 + \mathrm{Card}(E_2) \\
&\le n_0 + 2\bar{N}\gamma^{-1}(4M + 2\Lambda) = Q.
\end{aligned}$$

This completes the proof of Theorem 2.5.

2.7 Proof of Theorem 2.6

Set

$$\epsilon_0 = \epsilon(2\bar{N} + 1)^{-1}. \tag{2.145}$$

By (A1), there exists a positive number $\epsilon_1 < \epsilon_0$ such that the following property holds:

(P4) for each $i \in \{1, \dots, m\}$, each $z \in B(\theta, 3M+1) \cap C_i$, and each $x \in B(\theta, 3M+1)$ satisfying $d(x, C_i) \ge \epsilon_0$ we have

$$d(P_i(x), z) \le d(x, z) - \epsilon_1.$$

Set

$$\gamma_0 = 8^{-1}\epsilon_1(2\bar{N} + 1)^{-1}. \tag{2.146}$$

By (A1), there exists a positive number $\gamma < \min\{1, \gamma_0\}$ such that the following property holds:

(P5) for each $i \in \{1, \dots, m\}$, each $z \in B(\theta, 3M+1) \cap C_i$, and each $x \in B(\theta, 3M+1)$ satisfying $d(x, C_i) \ge \gamma_0/2$ we have

$$d(P_i(x), z) \le d(x, z) - \gamma.$$

Set

$$Q = \bar{N}(2M\gamma^{-1} + 1). \tag{2.147}$$

Assume that

$$r \in \mathcal{R}_{per}, \tag{2.148}$$

$\{x_i\}_{i=0}^{\infty} \subset X$,

$$x_0 \in B(\theta, M) \tag{2.149}$$

and

$$x_i = P_{r(i)}(x_{i-1}) \text{ for all natural numbers } i. \tag{2.150}$$

Set

$$P_r = P_{r(\bar{N})} \cdots P_{r(1)} = \prod_{i=1}^{\bar{N}} P_{r(i)}. \tag{2.151}$$

It follows from (2.105), (2.148), (2.150), and (2.151) that for each integer $i \geq 0$,

$$x_{(i+1)\bar{N}} = P_{r((i+1)\bar{N})} \cdots P_{r(i\bar{N}+1)}(x_{i\bar{N}}) = P_r(x_{i\bar{N}}). \tag{2.152}$$

Let n be a natural number and $\delta \in (0, 1)$. Assumption (A2) implies that there exists

$$z_\delta \in B(\theta, M_*) \tag{2.153}$$

such that

$$B(z_\delta, \delta) \cap C_i \neq \emptyset, \ \ i = 1, \dots, m. \tag{2.154}$$

In view of (2.154), for each $i \in \{1, \dots, m\}$ there exists

$$z_{\delta,i} \in C_i \cap B(z_\delta, \delta). \tag{2.155}$$

By (A3), (2.102), and (2.155), for each integer $i \in \{1, \dots, m\}$,

$$d(z_\delta, P_i(z_\delta)) \leq d(z_\delta, z_{\delta,i}) + d(P_i(z_{\delta,i}), P_i(z_\delta)) \leq 2d(z_\delta, z_{\delta,i}) \leq 2\delta. \tag{2.156}$$

It follows from (A3), (2.150), and (2.156) that for each integer $k \geq 0$,

$$d(z_\delta, x_{k+1}) = d(z_\delta, P_{r(k+1)}(x_k))$$

$$\leq d(z_\delta, P_{r(k+1)}(z_\delta)) + d(P_{r(k+1)}(z_\delta), P_{r(k+1)}(x_k)) \leq 2\delta + d(z_\delta, x_k). \tag{2.157}$$

Relations (2.149), (2.153), and (2.157) imply that for all integers $k = 0, \dots, n$,

$$d(z_\delta, x_k) \leq d(z_\delta, x_0) + 2\delta k \leq 2M + \delta n. \tag{2.158}$$

In view of (2.153) and (2.158), for all integers $k = 0, \dots, n$,

$$d(\theta, x_k) \le d(\theta, z_\delta) + d(z_\delta, x_k) \le 3M + \delta n. \tag{2.159}$$

Since δ is an arbitrary element of the interval $(0, 1)$ we conclude that

$$d(\theta, x_k) \le 3M \text{ for all integers } k \ge 0. \tag{2.160}$$

Set

$$E_0 = \{p \in \{0, 1, \dots\} : \max\{d(x_i, C_{r(i+1)}) : i = p\bar{N}, \dots, (p+1)\bar{N}-1\} \ge \gamma_0/2\}, \tag{2.161}$$

$$E_1 = \{0, 1, \dots\} \setminus E_0. \tag{2.162}$$

By (2.149), (2.153), and (2.157),

$$\begin{aligned}
2M &\ge d(z_\delta, x_0) \ge d(z_\delta, x_0) - d(z_\delta, x_{n\bar{N}}) \\
&= \sum_{k=0}^{n-1} (d(z_\delta, x_{k\bar{N}}) - d(z_\delta, x_{(k+1)\bar{N}})) \\
&= \sum_{k=0}^{n-1} [\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (d(z_\delta, x_j) - d(z_\delta, x_{j+1}))] \\
&= \sum \{ \sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (d(z_\delta, x_j) - d(z_\delta, x_{j+1})) : k \in E_1 \cap [0, n-1]\} \\
&= \sum \{ \sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (d(z_\delta, x_j) - d(z_\delta, x_{j+1})) : k \in E_0 \cap [0, n-1]\} \\
&\ge -2\delta n \bar{N} + \sum \{ \sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (d(z_\delta, x_j) - d(z_\delta, x_{j+1})) : k \in E_0 \cap [0, n-1]\}.
\end{aligned} \tag{2.163}$$

Assume that

$$k \in E_0. \tag{2.164}$$

By (2.161) and (2.164), there exists

$$j_0 \in \{k\bar{N}, \dots, (k+1)\bar{N} - 1\} \tag{2.165}$$

such that

$$d(x_{j_0}, C_{r(j_0+1)}) \geq \gamma_0/2. \tag{2.166}$$

Property (P5), (2.153), (2.155), (2.160), and (2.166) imply that

$$d(P_{r(j_0+1)}(x_{j_0}), z_{\delta,r(j_0+1)}) \leq d(x_{j_0}, z_{\delta,r(j_0+1)}) - \gamma. \tag{2.167}$$

It follows from (2.150), (2.155), and (2.167) that

$$\begin{aligned} d(x_{j_0+1}, z_\delta) &= d(P_{r(j_0+1)}(x_{j_0}), z_\delta) \\ &\leq d(P_{r(j_0+1)}(x_{j_0}), z_{\delta,r(j_0+1)}) + d(z_{\delta,r(j_0+1)}, z_\delta) \\ &\leq d(x_{j_0}, z_{\delta,r(j_0+1)}) - \gamma + \delta \leq d(x_{j_0}, z_\delta) + 2\delta - \gamma. \end{aligned} \tag{2.168}$$

By (2.157), (2.165), and (2.168),

$$\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (d(z_\delta, x_j) - d(z_\delta, x_{j+1})) \geq \gamma - 2\delta\bar{N}. \tag{2.169}$$

By (2.163) and (2.169),

$$2M + 2\delta n\bar{N} \geq (\gamma - 2\delta\bar{N})\mathrm{Card}(E_0 \cap [0, n-1]).$$

Since δ is any element of (0, 1) we conclude that

$$\mathrm{Card}(E_0 \cap [0, n-1]) \leq 2\gamma^{-1}M.$$

Since the inequality above holds for any natural number n we conclude that

$$\mathrm{Card}(E_0) \leq 2\gamma^{-1}M. \tag{2.170}$$

In view of (2.161) and (2.170), there exists an integer $q_0 \geq 0$ such that

$$q_0 \leq 2M\gamma^{-1} + 1, \tag{2.171}$$

$$d(x_j, C_{r(j+1)}) < \gamma_0/2, \ \ j = q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1. \tag{2.172}$$

Let

$$j \in \{q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1\}. \tag{2.173}$$

Relations (2.172) and (2.173) imply that there exists

$$\xi \in C_{r(j+1)} \tag{2.174}$$

such that

$$d(x_j, \xi) < \gamma_0/2. \tag{2.175}$$

It follows from (2.103), (2.150), (2.174), and (2.175) that

$$d(x_{j+1}, \xi) = d(P_{r(j+1)}(x_j), \xi) \leq d(x_j, \xi) < \gamma_0/2.$$

Together with (2.175) this implies that

$$d(x_j, x_{j+1}) < \gamma_0, \ \ j = q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1. \tag{2.176}$$

In view of (2.152) and (2.176),

$$d(x_{q_0\bar{N}}, P_r(x_{q_0\bar{N}})) = d(x_{q_0\bar{N}}, x_{(q_0+1)\bar{N}}) < \gamma_0\bar{N}. \tag{2.177}$$

Assumption (A3), (2.151), (2.152), and (2.177) imply that for each integer $q > q_0$,

$$\begin{aligned} d(x_{q\bar{N}}, x_{(q+1)\bar{N}}) &= d((P_r)^{q-q_0}(x_{q_0\bar{N}}), (P_r)^{q-q_0}(x_{(q_0+1)\bar{N}})) \\ &\leq d(x_{q_0\bar{N}}, x_{(q_0+1)\bar{N}}) < \gamma_0\bar{N}. \end{aligned} \tag{2.178}$$

Let $q \geq q_0$ be an integer. We show that

$$d(x_j, C_{r(j+1)}) \leq \epsilon_0, \ \ j = q\bar{N}, \dots, (q+1)\bar{N} - 1. \tag{2.179}$$

By (2.178),

$$d(x_{q\bar{N}}, x_{(q+1)\bar{N}}) < \gamma_0\bar{N}. \tag{2.180}$$

Assume that (2.179) does not hold. Then there exists

$$j_0 \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\} \tag{2.181}$$

such that

$$d(x_{j_0}, C_{r(j_0+1)}) > \epsilon_0. \tag{2.182}$$

It follows from (2.150), (2.153), (2.155), (2.159), (2.182), and property (P4) that

$$d(x_{j_0+1}, z_{\delta,r(j_0+1)}) = d(P_{r(j_0+1)}(x_{j_0}), z_{\delta,r(j_0+1)}) \leq d(x_{j_0}, z_{\delta,r(j_0+1)}) - \epsilon_1. \tag{2.183}$$

In view of (2.155) and (2.183),

$$d(z_\delta, x_{j_0+1}) \le d(z_\delta, z_{\delta,r(j_0+1)}) + d(z_{\delta,r(j_0+1)}, x_{j_0+1})$$

$$\le \delta + d(z_{\delta,r(j_0+1)}, x_{j_0+1}) \le d(x_{j_0}, z_{\delta,r(j_0+1)}) - \epsilon_1 + \delta \le d(z_\delta, x_{j_0}) - \epsilon_1 + 2\delta. \tag{2.184}$$

By (2.157), (2.180), (2.181), and (2.184),

$$\gamma_0 \bar{N} > d(x_{q\bar{N}}, x_{(q+1)\bar{N}}) \ge d(z_\delta, x_{q\bar{N}}) - d(z_\delta, x_{(q+1)\bar{N}})$$

$$= \sum_{j=q\bar{N}}^{(q+1)\bar{N}-1} (d(z_\delta, x_j) - d(z_\delta, x_{j+1})) \ge \epsilon_1 - 2\delta\bar{N}.$$

Since δ is any element of the interval (0, 1) we conclude that

$$\epsilon_1 \le \gamma_0 \bar{N}.$$

This contradicts (2.146). The contradiction we have reached proves (2.179).

Let $j \in \{q\bar{N}, \ldots, (q+1)\bar{N} - 1\}$ and $\kappa > 0$. In view of (2.179),

$$d(x_j, C_{r(j+1)}) \le \epsilon_0$$

and there exists

$$\xi \in C_{r(j+1)} \tag{2.185}$$

such that

$$d(x_j, \xi) < \epsilon_0 + \kappa. \tag{2.186}$$

Assumption (A3), (2.102), (2.150), (2.185), and (2.186) imply that

$$d(x_{j+1}, \xi) = d(P_{r(j+1)}(x_j), P_{r(j+1)}(\xi)) \le d(x_j, \xi) < \epsilon_0 + \kappa$$

and

$$2\kappa + 2\epsilon_0 > d(x_j, x_{j+1}), \;\; j = q\bar{N}, \ldots, (q+1)\bar{N} - 1.$$

Since κ is any positive number we conclude that

$$2\epsilon_0 \ge d(x_j, x_{j+1}), \;\; j = q\bar{N}, \ldots, (q+1)\bar{N} - 1. \tag{2.187}$$

In view of (2.187), for each $j_1, j_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$d(x_{j_1}, x_{j_2}) \le 2\epsilon_0 \bar{N}. \tag{2.188}$$

Let

$$k \in \{q\bar{N}, \dots, (q+1)\bar{N}\},\ \ s \in \{1, \dots, m\}. \tag{2.189}$$

By (2.104), (2.148), and (2.189), there exists

$$j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\} \tag{2.190}$$

such that

$$r(j+1) = s. \tag{2.191}$$

By (2.179), (2.190), and (2.191),

$$d(x_j, C_s) \le \epsilon_0.$$

Together with (2.145) and (2.188)–(2.190) this implies that

$$d(x_k, C_s) \le d(x_k, x_j) + d(x_j, C_s) \le \epsilon_0(2\bar{N} + 1) = \epsilon$$

and

$$\max\{d(x_k, C_s) :\ s = 1, \dots, m\} \le \epsilon$$

for all integers $k \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$ and all integers $q \ge q_0$. Since $Q = q_0\bar{N}$ this completes the proof of Theorem 2.6.

2.8 Auxiliary Results

Let (Y, ρ) be a metric space. Denote by $\mathfrak{M}_Y$ the set of all mappings $T : Y \to Y$ such that

$$\rho(T(x), T(y)) \le \rho(x, y) \text{ for all } x, y \in Y. \tag{2.192}$$

For each $y \in Y$ and each $r > 0$ set

$$B(y, r) = \{z \in Y :\ \rho(y, z) \le r\}.$$

Proposition 2.8 *Let $n \ge 1$ be an integer, $\delta > 0$,*

$$\{T_i\}_{i=1}^n \subset \mathfrak{M}_Y,\ \{x_i\}_{i=0}^n,\ \{y_i\}_{i=0}^n \subset Y,$$

$$x_0 = y_0 \tag{2.193}$$

and let for all integers $i = 1, \dots, n$,

$$y_i = T_i(y_{i-1}),\ \rho(x_i, T_i(x_{i-1})) \le \delta. \tag{2.194}$$

Then for all integers $i = 0, \dots, n$,

$$\rho(x_i, y_i) \le i\delta. \tag{2.195}$$

Proof By (2.193), inequality (2.195) holds for $i = 0$. Assume that $i < n$ is a nonnegative integer and that (2.195) holds. It follows from (2.192), (2.194), and (2.195) that

$$\rho(x_{i+1}, y_{i+1}) \le \rho(x_{i+1}, T_{i+1}(x_i)) + \rho(T_{i+1}(x_i), T_{i+1}(y_i))$$

$$\le \delta + \rho(x_i, y_i) \le (i+1)\delta.$$

Therefore (2.195) holds for all $i = 0, \dots, n$. Proposition 2.8 is proved.

Theorem 2.9 *Let N be a natural number, $\mathfrak{A}$ be a set of mappings $S : \{1, 2, \dots\} \to \mathfrak{M}_Y$ such that*

$$S(i + N) = S(i) \text{ for all integers } i \ge 1 \tag{2.196}$$

and let $F \subset Y$ be a nonempty bounded set. Assume that for each $M > 0$ there exists an integer $Q > 0$ such that the following property holds:

(P6) for each $S \in \mathfrak{A}$ and each sequence $\{x_i\}_{i=0}^\infty \subset Y$ which satisfies

$$x_0 \in B(\theta, M),$$

$$x_{i+1} = S(i+1)(x_i) \text{ for all integers } i \ge 0$$

the inclusion $x_i \in F$ holds for all integers $i \ge Q$.

Let $M > 1, \epsilon \in (0, 1)$, an integer $Q > 0$ be such that property (P6) holds and let

$$\delta = \epsilon(Q(2N+1))^{-1}.$$

Then for each $S \in \mathfrak{A}$ *and each sequence* $\{x_i\}_{i=0}^{\infty} \subset Y$ *which satisfies*

$$x_0 \in B(\theta, M), \tag{2.197}$$

$$\rho(x_{i+1}, S(i+1)(x_i)) \le \delta \text{ for all integers } i \ge 0 \tag{2.198}$$

the relation

$$B(x_i, \epsilon) \cap F \ne \emptyset \tag{2.199}$$

holds for all integers $i \ge Q$.

Proof We may assume without loss of generality that

$$F \subset B(\theta, M-1). \tag{2.200}$$

Fix

$$\delta = \epsilon(Q(2N+1))^{-1}. \tag{2.201}$$

Assume that

$$S \in \mathfrak{A} \tag{2.202}$$

and that a sequence $\{x_i\}_{i=0}^{\infty} \subset Y$ satisfy (2.197) and (2.198). We show that (2.199) holds.

Assume that n is a nonnegative integer and that

$$x_{nN} \in B(\theta, M). \tag{2.203}$$

Consider a sequence $\{y_i\}_{i=nN}^{\infty} \subset Y$ such that

$$y_{nN} = x_{nN}, \tag{2.204}$$

$$y_{i+1} = S(i+1)y_i \text{ for all integers } i \ge nN. \tag{2.205}$$

Property (P6), the choice of Q, (2.196), and (2.202)–(2.205) imply that

$$y_i \in F \text{ for all integers } i \ge Q + nN. \tag{2.206}$$

Proposition 2.8, (2.192), (2.196), (2.197), (2.198), and (2.202)–(2.205) imply that for each integer $i \ge n\bar{N}$,

$$\rho(x_i, y_i) \le \delta(i - nN). \tag{2.207}$$

By (2.201) and (2.207), for each integer $i \in \{nN + Q, \dots, nN + Q + 2QN\}$,

$$\rho(x_i, y_i) \le \delta(Q(2N + 1)) \le \epsilon.$$

Together with (2.206) this implies that for all integers

$$i \in \{nN + Q, \dots, nN + Q + 2QN\},$$

we have

$$B(x_i, \epsilon) \cap F \neq \emptyset.$$

Thus we have shown that the following property holds:

(P7) If n is a nonnegative integer and $x_{nN} \in B(\theta, M)$, then

$$B(x_i, \epsilon) \cap F \neq \emptyset, \ i \in \{nN + Q, \dots, nN + Q + 2QN\}. \tag{2.208}$$

Property (P7) and (2.197) imply that

$$B(x_i, \epsilon) \cap F \neq \emptyset, \ i = Q, \dots, Q + 2QN. \tag{2.209}$$

Assume that an integer $q \ge Q$ and

$$B(x_i, \epsilon) \cap F \neq \emptyset, \ i = Q, \dots, Q + 2qN. \tag{2.210}$$

(Note that in view of (2.109), our assumption holds for $q = Q$.) By (2.200) and (2.210),

$$B(x_{2qN}, \epsilon) \cap F \neq \emptyset,$$

$$x_{2qN} \in B(\theta, M). \tag{2.211}$$

It follows from (2.211) and property (P7) applied with $n = 2q$ that

$$B(x_i, \epsilon) \cap F \neq \emptyset, \ i = 2qN + Q, \dots, Q + 2qN + 2QN.$$

In view of the relation above and (2.210),

$$B(x_i, \epsilon) \cap F \neq \emptyset, \ i = Q, \dots, Q + 2qN + 2QN.$$

Thus by induction we showed that

$$B(x_i, \epsilon) \cap F \neq \emptyset \text{ for all integers } i \ge Q.$$

Theorem 2.9 is proved.

2.9 Proof of Theorem 2.7

We may assume that

$$\epsilon_0 < \min\{1, r_0\}/4.$$

We deduce Theorem 2.7 from Theorems 2.6 and 2.9. Let

$$(Y, \rho) = (X, d),\ N = \bar{N},\ \mathfrak{A} = \{i \to P_{r(i)},\ i = 1, 2, \dots : r \in \mathcal{R}_{per}\},$$

$$F = \{x \in X : \max\{d(x, C_s) : s = 1, \dots, m\} \le \epsilon_0/4\}. \tag{2.212}$$

In view of Theorem 2.6, for each $M > 0$ there exists an integer $Q > 0$ such that property (P6) holds. Hence Theorem 2.9 implies that there exist $\delta > 0$ and an integer $Q > 0$ such that for each $r \in \mathcal{R}_{per}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies

$$x_0 \in B(\theta, M), \tag{2.213}$$

$$\rho(x_{i+1}, P_{r(i+1)}(x_i)) \le \delta \text{ for all integers } i \ge 0 \tag{2.214}$$

we have

$$B(x_i, \epsilon_0/4) \cap F \ne \emptyset \text{ for all integers } i \ge Q. \tag{2.215}$$

Assume that $r \in \mathcal{R}_{per}$ and that a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies (2.213) and (2.214). Then (2.215) is true.

Assume that an integer $i \ge Q$. By (2.215), there exists

$$\xi \in F \cap B(x_i, \epsilon_0/4).$$

In view of (2.212), for all $s = 1, \dots, m$,

$$d(x_i, C_s) \le d(x_i, \xi) + d(\xi, C_i) \le \epsilon_0/4 + \epsilon_0/4,$$

$$\max\{d(x_i, C_s) : s = 1, \dots, m\} < \epsilon_0 \text{ for all integers } i \ge Q.$$

Theorem 2.7 is proved.

2.10 The Third Problem

Let (X, d) be a metric space. Recall that for each $x \in X$ and each $r > 0$,

$$B(x, r) = \{y \in X : d(x, y) \le r\}$$

and that for each $x \in X$ and each nonempty set $E \subset X$

$$d(x, E) = \inf\{d(x, y) : \; y \in E\}.$$

Fix $\theta \in X$. Let m be a natural number and let $P_i : X \to X$, $i = 1, \dots, m$ be self-mappings of the space X. Suppose that for every $i \in \{1, \dots, m\}$,

$$F_i := \mathrm{Fix}(P_i) := \{z \in X : \; P_i(z) = z\} \neq \emptyset. \tag{2.216}$$

For every $\epsilon > 0$ and every $i \in \{1, \dots, m\}$ set

$$F_\epsilon(P_i) = \{x \in X : \; d(x, P_i(x)) \leq \epsilon\}, \tag{2.217}$$

$$\tilde{F}_\epsilon(P_i) = \{y \in X : \; d(y, F_\epsilon(P_i)) \leq \epsilon\}, \tag{2.218}$$

$$F_\epsilon = \cap_{i=1}^m F_\epsilon(P_i), \;\; \tilde{F}_\epsilon = \cap_{i=1}^m \tilde{F}_\epsilon(P_i). \tag{2.219}$$

Let $M_* > 1$ and suppose that the following properties hold.

(A4) For each $M > 0$ and each $\gamma > 0$ there exists $\delta > 0$ such that for each $i \in \{1, \dots, m\}$, each

$$z \in B(\theta, M) \cap \mathrm{Fix} P_i$$

and each $x \in B(\theta, M)$ satisfying $d(x, P_i(x)) \geq \gamma$, the inequality

$$d(P_i(x), z) \leq d(x, z) - \delta$$

is true.

(A5) For each $\delta > 0$ there exists $z_\delta \in B(\theta, M_*)$ such that

$$B(z_\delta, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m.$$

In view of (A4) and (2.216), for each $i = 1, \dots, m$,

$$d(P_i(x), z) \leq d(x, z) \text{ for each } x \in X \text{ and each } z \in \mathrm{Fix}(P_i). \tag{2.220}$$

Fix a natural number $\bar{N} \geq m$. Denote by $\mathcal{R}$ the set of all mappings $r : \{1, 2, \dots\} \to \{1, \dots, m\}$ such that for each number j,

$$\{1, \dots, m\} \subset \{r(j), \dots, r(j + \bar{N} - 1)\} \tag{2.221}$$

and denote by $\mathcal{R}_{per}$ the set of all $r \in \mathcal{R}$ such that for each integer $i \geq 1$,

$$r(i + \bar{N}) = r(i). \tag{2.222}$$

In this chapter we prove the following three results: Theorem 2.10 which shows that the inexact iterative method generates approximate solutions if perturbations are summable, Theorem 2.11 which establishes that the exact iterative method generates approximate solutions, and Theorem 2.12 which demonstrates that the inexact iterative method generates approximate solutions if the perturbations are small enough.

Theorem 2.10 *Let $M \ge M_*$, ϵ be a positive number and let a sequence $\{\epsilon_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy*

$$\Lambda := \sum_{i=1}^{\infty} \epsilon_i < \infty. \tag{2.223}$$

Then there exists a constant $Q > 0$ such that for each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies

$$x_0 \in B(\theta, M),$$

$$d(x_i, P_{r(i)}(x_{i-1})) \le \epsilon_i \text{ for all natural numbers } i$$

the inequality

$$Card(\{i \in \{0, 1, \dots\} : \ x_i \notin \tilde{F}_\epsilon\}) \le Q$$

holds.

Theorem 2.11 *Assume that the following property holds:*

(A6) $d(P_i(x), P_i(y)) \le d(x, y)$ for all $x, y \in X$ and all $i = 1, \dots, m$.

Let $M \ge M_*$, $\epsilon > 0$. Then there exists a constant $Q > 0$ such that for each $r \in \mathcal{R}_{per}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies

$$x_0 \in B(\theta, M),$$

$$x_i = P_{r(i)}(x_{i-1}) \text{ for all natural numbers } i$$

the inclusion $x_i \in F_\epsilon$ holds for all integer $i \ge Q$.

Theorem 2.12 *Assume that (A6) holds. Let $M \ge M_*$, $r_0 > 0$,*

$$F_{r_0} \subset B(\theta, M)$$

and $\epsilon_0 > 0$. Then there exist $Q, \delta > 0$ such that for each $r \in \mathcal{R}_{per}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies

$$x_0 \in B(\theta, M),$$

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \delta \textit{ for all natural numbers } i$$

the inclusion $x_i \in F_{\epsilon_0}$ *holds for all integer* $i \geq Q$.

2.11 Proof of Theorem 2.10

We may assume that $\epsilon < 1$.

Set

$$\gamma_0 = \epsilon(2\bar{N} + 1)^{-1}. \tag{2.224}$$

By (A4), there exists a positive number $\gamma < \gamma_0$ such that the following property holds:

(P8) for each $i \in \{1, \dots, m\}$, each $z \in B(\theta, 3M + 1 + \Lambda) \cap \mathrm{Fix}(P_i)$, and each $x \in B(\theta, 3M + 1 + \Lambda)$ satisfying $d(x, P_i(x)) \geq \gamma_0/2$ we have

$$d(P_i(x), z) \leq d(x, z) - \gamma.$$

By (2.223), there exists a natural number n_0 such that

$$\epsilon_i < \gamma/4 \text{ for all integers } i \geq n_0. \tag{2.225}$$

Set

$$Q = n_0 + 2\bar{N}\gamma^{-1}(4M + 2\Lambda). \tag{2.226}$$

Assume that $\{x_i\}_{i=0}^{\infty} \subset X$,

$$r \in \mathcal{R},\ \ x_0 \in B(\theta, M) \tag{2.227}$$

and

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \epsilon_i \text{ for all natural numbers } i. \tag{2.228}$$

Set

$$\epsilon_0 = 0. \tag{2.229}$$

Let n be a natural number and $\delta \in (0, 1)$. By (A5), there exists

$$z_\delta \in B(\theta, M_*) \tag{2.230}$$

such that

$$B(z_\delta, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m. \tag{2.231}$$

By (2.231), for each $i \in \{1, \dots\}$ there exists

$$z_{\delta,i} \in \mathrm{Fix}(P_i) \tag{2.232}$$

such that

$$d(z_\delta, z_{\delta,i}) \leq \delta. \tag{2.233}$$

It follows from (2.216), (2.220), (2.228), (2.232), and (2.233) that for each integer $i \geq 0$,

$$\begin{aligned} d(z_\delta, x_{i+1}) &\leq d(z_\delta, z_{\delta,r(i+1)}) + d(z_{\delta,r(i+1)}, x_{i+1}) \\ &\leq \delta + d(z_{\delta,r(i+1)}, P_{r(i+1)}(x_i)) + d(P_{r(i+1)}(x_i), x_{i+1}) \\ &\leq \delta + \epsilon_{i+1} + d(z_{\delta,r(i+1)}, x_i) \\ &\leq 2\delta + \epsilon_{i+1} + d(z_\delta, x_i). \end{aligned} \tag{2.234}$$

By induction we show that for all integers $k = 0, \dots, n$,

$$d(z_\delta, x_k) \leq 2M + \sum_{i=0}^{k} \epsilon_i + 2\delta k. \tag{2.235}$$

In view of (2.227), (2.229), and (2.230), inequality (2.235) holds for $k = 0$. Assume that a nonnegative integer $k < n$ and that (2.235) holds. It follows from (2.234) and (2.235) that

$$\begin{aligned} d(z_\delta, x_{k+1}) &\leq 2\delta + \epsilon_{k+1} + d(z_\delta, x_k) \\ &\leq 2M + \sum_{i=0}^{k+1} \epsilon_i + 2\delta(k+1). \end{aligned}$$

Thus (2.235) holds for all $k = 0, \dots, n$. It follows from (2.230) and (2.235) that for all $k = 0, \dots, n$,

$$d(\theta, x_k) \leq d(\theta, z_\delta) + d(z_\delta, x_k) \leq 3M + \sum_{i=0}^{n} \epsilon_i + 2\delta n.$$

Since δ is any element of the interval (0, 1) we conclude (see (2.223)) that

$$d(\theta, x_k) \leq 3M + \Lambda \text{ for all integers } k \geq 0. \tag{2.236}$$

Set

$$E_0 = \{i \in \{n_0, n_0 + 1, \dots\} : \ d(x_i, P_{r(i+1)}(x_i)) \geq \gamma_0/2\}, \tag{2.237}$$

$$E_1 = \{n_0, n_0 + 1, \dots\} \setminus E_0.$$

In view of (2.234), for all integers $i \geq 0$,

$$d(z_\delta, x_{i+1}) \leq d(z_\delta, x_i) + 2\delta + \epsilon_{i+1}. \tag{2.238}$$

Let

$$i \in E_0. \tag{2.239}$$

Property (P8), (2.230), (2.232), (2.233), (2.236), (2.237), and (2.239) imply that

$$d(P_{r(i+1)}(x_i), z_{\delta,r(i+1)}) \leq d(x_i, z_{\delta,r(i+1)}) - \gamma. \tag{2.240}$$

It follows from (2.233) and (2.240) that

$$\begin{aligned} d(P_{r(i+1)}(x_i), z_\delta) &\leq d(P_{r(i+1)}(x_i), z_{\delta,r(i+1)}) + d(z_{\delta,r(i+1)}, z_\delta) \\ &\leq \delta + d(x_i, z_{\delta,r(i+1)}) - \gamma \\ &\leq d(x_i, z_\delta) + 2\delta - \gamma. \end{aligned} \tag{2.241}$$

By (2.225), (2.228), (2.237), (2.239), and (2.241),

$$\begin{aligned} d(x_{i+1}, z_\delta) &\leq d(x_{i+1}, P_{r(i+1)}(x_i)) + d(P_{r(i+1)}(x_i), z_\delta) \\ &\leq 2\delta + d(x_i, z_\delta) - \gamma + \epsilon_{i+1} \\ &\leq d(x_i, z_\delta) + 2\delta - \gamma/2. \end{aligned} \tag{2.242}$$

Let $n > n_0$ be an integer. By (2.230), (2.236)–(2.238), and (2.242),

$$\begin{aligned} 4M + \Lambda \geq d(z_\delta, x_{n_0}) &\geq d(z_\delta, x_{n_0}) - d(z_\delta, x_n) \\ &= \sum_{i=n_0}^{n-1} (d(z_\delta, x_i) - d(z_\delta, x_{i+1})) \end{aligned}$$

$$= \sum\{d(z_\delta, x_i) - d(z_\delta, x_{i+1}) : i \in E_1 \cap [0, n-1]\}$$

$$= \sum\{d(z_\delta, x_i) - d(z_\delta, x_{i+1}) : i \in E_0 \cap [0, n-1]\}$$

$$\geq \sum\{-2\delta - \epsilon_{i+1} : i \in E_1 \cap [0, n-1]\}$$

$$+ (\gamma/2 - 2\delta)\mathrm{Card}(E_0 \cap [0, n-1]). \quad (2.243)$$

In view of (2.223) and (2.243),

$$(\gamma/2 - 2\delta)\mathrm{Card}(E_0 \cap [0, n-1]) \leq 4M + 2\Lambda + 2\delta n.$$

Since δ is any element of the interval (0, 1) we conclude that

$$\mathrm{Card}(E_0 \cap [0, n-1]) \leq 2\gamma^{-1}(4M + 2\Lambda).$$

Since n is any integer satisfying $n > n_0$ we conclude that

$$\mathrm{Card}(E_0) \leq 2\gamma^{-1}(4M + 2\Lambda). \quad (2.244)$$

Set

$$E_2 = \{k \in \{n_0, n_0 + 1, \dots\} : [k, k + \bar{N} - 1] \cap E_0 \neq \emptyset\}. \quad (2.245)$$

By (2.244) and (2.245),

$$\mathrm{Card}(E_2) \leq \bar{N}\mathrm{Card}(E_0)$$

$$\leq 2\bar{N}\gamma^{-1}(4M + 2\Lambda). \quad (2.246)$$

Let a nonnegative integer p satisfies

$$p \geq n_0 \text{ and } p \notin E_2. \quad (2.247)$$

Then in view of (2.237), (2.245), and (2.247),

$$[p, p + \bar{N} - 1] \cap E_0 = \emptyset$$

and for each $k \in \{p, \dots, p + \bar{N} - 1\}$,

$$d(x_k, P_{r(k+1)}(x_k)) < \gamma_0/2. \quad (2.248)$$

Let

$$k \in \{p, \dots, p + \bar{N} - 1\}. \tag{2.249}$$

By (2.225), (2.228), (2.247), (2.248), and (2.249),

$$d(x_k, x_{k+1}) \le d(x_k, P_{r(k+1)}(x_k)) + d(P_{r(k+1)}(x_k), x_{k+1})$$

$$< \gamma_0/2 + \epsilon_{k+1} < 3\gamma_0/4.$$

Thus

$$d(x_k, x_{k+1}) < 3\gamma_0/4 \text{ for all } k = p, \dots, p + \bar{N} - 1. \tag{2.250}$$

In view of (2.250), for all integers $i_1, i_2 \in \{p, \dots, p + \bar{N}\}$,

$$d(x_{i_1}, x_{i_2}) \le 3\bar{N}\gamma_0/4. \tag{2.251}$$

Let $s \in \{1, \dots, m\}$. By (2.221) and (2.227), there exists

$$k \in \{p, \dots, p + \bar{N} - 1\} \tag{2.252}$$

such that

$$r(k + 1) = s. \tag{2.253}$$

It follows from (2.249), (2.252), and (2.253) that

$$d(x_k, P_s(x_k)) < \gamma_0/2. \tag{2.254}$$

By (2.224), (2.251), (2.252), and (2.254), for each $i \in \{p, \dots, p + \bar{N}\}$,

$$x_i \in \tilde{F}_{\bar{N}\gamma_0}(P_s) \subset \tilde{F}_\epsilon(P_s), \;\; s = 1, \dots, m$$

and

$$x_i \in \tilde{F}_\epsilon.$$

Thus for all integers $p \ge n_0$ satisfying $p \notin E_2$, we have

$$x_p \in \tilde{F}_\epsilon$$

and in view of (2.226) and (2.246),

$$\mathrm{Card}(\{i \in \{0, 1, \dots\} : \; x_i \notin \tilde{F}_\epsilon\})$$

$$\leq n_0 + \text{Card}(E_2)$$

$$\leq n_0 + 2\bar{N}\gamma^{-1}(4M + 2\Lambda) = Q.$$

This completes the proof of Theorem 2.10.

2.12 Proof of Theorem 2.11

Set

$$\epsilon_0 = \epsilon(2\bar{N} + 1)^{-1}. \tag{2.255}$$

By (A4), there exists a positive number $\epsilon_1 < \epsilon_0$ such that the following property holds:

(P9) for each $i \in \{1, \dots, m\}$, each $z \in B(\theta, 3M + 1) \cap \text{Fix}(P_i)$, and each $x \in B(\theta, 3M + 1)$ satisfying $d(x, P_i(x)) \geq \epsilon_0$ we have

$$d(P_i(x), z) \leq d(x, z) - \epsilon_1.$$

Set

$$\gamma_0 = 8^{-1}\epsilon_1(2\bar{N} + 1)^{-1}. \tag{2.256}$$

By (A4), there exists a positive number $\gamma < \min\{1, \gamma_0\}$ such that the following property holds:

(P10) for each $i \in \{1, \dots, m\}$, each $z \in B(\theta, 3M + 1) \cap \text{Fix}(P_i)$, and each $x \in B(\theta, 3M + 1)$ satisfying $d(x, P_i(x)) \geq \gamma_0/2$ we have

$$d(P_i(x), z) \leq d(x, z) - \gamma.$$

Set

$$Q = (2M\gamma^{-1} + 1)\bar{N}. \tag{2.257}$$

Assume that

$$r \in \mathcal{R}_{per}, \tag{2.258}$$

$\{x_i\}_{i=0}^{\infty} \subset X$,

$$x_0 \in B(\theta, M) \tag{2.259}$$

and

$$x_i = P_{r(i)}(x_{i-1}) \text{ for all natural numbers } i. \tag{2.260}$$

Set

$$P_r = P_{r(\bar{N})} \cdots P_{r(1)} = \prod_{i=1}^{\bar{N}} P_{r(i)}. \tag{2.261}$$

It follows from (2.222) and (2.258)–(2.260) that for each integer $i \geq 0$,

$$x_{(i+1)\bar{N}} = P_{r((i+1)\bar{N})} \cdots P_{r(i\bar{N}+1)}(x_{i\bar{N}}) = P_r(x_{i\bar{N}}). \tag{2.262}$$

Let n be a natural number and $\delta \in (0, 1)$. Assumption (A5) implies that there exists

$$z_\delta \in B(\theta, M_*) \tag{2.263}$$

such that

$$B(z_\delta, \delta) \cap \text{Fix}(P_i) \neq \emptyset, \ \ i = 1, \dots, m. \tag{2.264}$$

In view of (2.264), for each $i \in \{1, \dots, m\}$ there exists

$$z_{\delta,i} \in \text{Fix}(P_i) \cap B(z_\delta, \delta). \tag{2.265}$$

(A6), (2.216), and (2.265) imply that for each $i = 1, \dots, m$,

$$d(z_\delta, P_i(z_\delta)) \leq d(z_\delta, z_{\delta,i}) + d(P_i(z_{\delta,i}), P_i(z_\delta))$$

$$\leq 2d(z_\delta, z_{\delta,i}) \leq 2\delta. \tag{2.266}$$

It follows from (A6), (2.260), and (2.266) that for each integer $k \geq 0$,

$$d(z_\delta, x_{k+1}) = d(z_\delta, P_{r(k+1)}(x_k))$$

$$\leq d(z_\delta, P_{r(k+1)}(z_\delta)) + d(P_{r(k+1)}(z_\delta), P_{r(k+1)}(x_k)) \leq 2\delta + d(z_\delta, x_k). \tag{2.267}$$

Relations (2.259), (2.263), and (2.267) imply that for all integers $k = 0, \dots, n$,

$$d(z_\delta, x_k) \leq d(z_\delta, x_0) + 2\delta k \leq 2M + \delta n. \tag{2.268}$$

In view of (2.263) and (2.268), for all integers $k = 0, \dots, n$,

$$d(\theta, x_k) \leq d(\theta, z_\delta) + d(z_\delta, x_k) \leq 3M + \delta n.$$

Since δ is an arbitrary element of the interval (0, 1) we conclude that

$$d(\theta, x_k) \leq 3M \text{ for all integers } k \geq 0. \tag{2.269}$$

Set

$$E_0 = \{p \in \{0, 1, \dots\} :$$
$$\max\{d(x_i, P_{r(i+1)}(x_i)) : \ i = p\bar{N}, \dots, (p+1)\bar{N} - 1\} \geq \gamma_0/2\}, \tag{2.270}$$
$$E_1 = \{0, 1, \dots\} \setminus E_0.$$

By (2.259), (2.263), (2.267), and (2.270),

$$\begin{aligned}
2M &\geq d(z_\delta, x_0) \geq d(z_\delta, x_0) - d(z_\delta, x_{n\bar{N}}) \\
&= \sum_{k=0}^{n-1} (d(z_\delta, x_{k\bar{N}}) - d(z_\delta, x_{(k+1)\bar{N}})) \\
&= \sum_{k=0}^{n-1} [\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (d(z_\delta, x_j) - d(z_\delta, x_{j+1}))] \\
&= \sum \{\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (d(z_\delta, x_j) - d(z_\delta, x_{j+1})) : \ k \in E_1 \cap [0, n-1]\} \\
&+ \sum \{\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (d(z_\delta, x_j) - d(z_\delta, x_{j+1})) : \ k \in E_0 \cap [0, n-1]\} \\
&\geq -2\delta n\bar{N} + \sum \{\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (d(z_\delta, x_j) - d(z_\delta, x_{j+1})) : \ k \in E_0 \cap [0, n-1]\}.
\end{aligned} \tag{2.271}$$

Assume that

$$k \in E_0. \tag{2.272}$$

By (2.270) and (2.272), there exists

$$j_0 \in \{k\bar{N}, \dots, (k+1)\bar{N} - 1\} \tag{2.273}$$

such that

$$d(x_{j_0}, P_{r(j_0+1)}(x_{j_0})) \geq \gamma_0/2. \tag{2.274}$$

Property (P10), (2.260), (2.263), (2.265), (2.269), and (2.273) imply that

$$d(x_{j_0+1}, z_{\delta,r(j_0+1)}) = d(P_{r(j_0+1)}(x_{j_0}), z_{\delta,r(j_0+1)}) \leq d(x_{j_0}, z_{\delta,r(j_0+1)}) - \gamma. \tag{2.275}$$

By (2.265) and (2.275),

$$d(x_{j_0+1}, z_\delta) \leq d(z_\delta, z_{\delta,r(j_0+1)}) + d(z_{\delta,r(j_0+1)}, x_{j_0+1})$$

$$\leq d(x_{j_0}, z_{\delta,r(j_0+1)}) - \gamma + \delta \leq d(x_{j_0}, z_\delta) + 2\delta - \gamma. \tag{2.276}$$

It follows from (2.267), (2.273), (2.274), and (2.276) that

$$\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (d(z_\delta, x_j) - d(z_\delta, x_{j+1})) \geq \gamma - 2\delta\bar{N}. \tag{2.277}$$

By (2.271), (2.272), and (2.277),

$$2M + 2\delta n\bar{N} \geq (\gamma - 2\delta\bar{N})\mathrm{Card}(E_0 \cap [0, n-1]).$$

Since δ is any element of (0, 1) we conclude that

$$\mathrm{Card}(E_0 \cap [0, n-1]) \leq 2\gamma^{-1}M.$$

Since the inequality above holds for any natural number n we conclude that

$$\mathrm{Card}(E_0) \leq 2\gamma^{-1}M. \tag{2.278}$$

In view of (2.260), (2.270), and (2.278), there exists an integer $q_0 \geq 0$ such that

$$q_0 \leq 2M\gamma^{-1} + 1, \tag{2.279}$$

$$d(x_i, x_{i+1}) = d(x_i, P_{r(i+1)}(x_i)) < \gamma_0/2,\ i = q_0\bar{N}, \dots, (q_0+1)\bar{N}-1. \tag{2.280}$$

In view of (2.262) and (2.280),

$$d(x_{q_0\bar{N}}, P_r(x_{q_0\bar{N}})) = d(x_{q_0\bar{N}}, x_{(q_0+1)\bar{N}}) < 2^{-1}\gamma_0\bar{N}. \tag{2.281}$$

Assumption (A6), (2.260)–(2.262), and (2.281) imply that for each integer $q > q_0$,

$$d(x_{q\bar{N}}, x_{(q+1)\bar{N}}) = d((P_r)^{q-q_0}(x_{q_0\bar{N}}), (P_r)^{q-q_0}(x_{(q_0+1)\bar{N}}))$$
$$\leq d(x_{q_0\bar{N}}, x_{(q_0+1)\bar{N}}) < 2^{-1}\gamma_0\bar{N}. \tag{2.282}$$

Let $q \geq q_0$ be an integer. We show that

$$d(x_j, x_{j+1}) \leq \epsilon_0, \ j = q\bar{N}, \dots, (q+1)\bar{N} - 1. \tag{2.283}$$

By (2.282),

$$d(x_{q\bar{N}}, x_{(q+1)\bar{N}}) < 2^{-1}\gamma_0\bar{N}. \tag{2.284}$$

Assume that (2.283) does not hold. Then there exists

$$j_0 \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\} \tag{2.285}$$

such that

$$d(x_{j_0}, x_{j_0+1}) > \epsilon_0. \tag{2.286}$$

It follows from (2.260) and (2.286) that

$$d(x_{j_0}, P_{r(j_0+1)}(x_{j_0})) > \epsilon_0. \tag{2.287}$$

Property (P9), (2.260), (2.263), (2.265), (2.269), and (2.287) imply that

$$d(x_{j_0+1}, z_{\delta,r(j_0+1)}) = d(P_{r(j_0+1)}(x_{j_0}), z_{\delta,r(j_0+1)}) \leq d(x_{j_0}, z_{\delta,r(j_0+1)}) - \epsilon_1. \tag{2.288}$$

In view of (2.265) and (2.288),

$$d(z_\delta, x_{j_0+1}) \leq \delta + d(z_{\delta,r(j_0+1)}, x_{j_0+1})$$
$$\leq \delta + d(x_{j_0}, z_{\delta,r(j_0+1)}) - \epsilon_1 \leq d(z_\delta, x_{j_0}) - \epsilon_1 + 2\delta. \tag{2.289}$$

By (2.267), (2.284), and (2.289),

$$2^{-1}\gamma_0\bar{N} > d(x_{q\bar{N}}, x_{(q+1)\bar{N}}) \geq d(z_\delta, x_{q\bar{N}}) - d(z_\delta, x_{(q+1)\bar{N}})$$
$$= \sum_{j=q\bar{N}}^{(q+1)\bar{N}-1} (d(z_\delta, x_j) - d(z_\delta, x_{j+1})) \geq \epsilon_1 - 2\delta\bar{N}.$$

Since δ is any element of the interval (0, 1) we conclude that

$$\epsilon_1 \le 2^{-1}\gamma_0 \bar{N}.$$

This contradicts (2.256). The contradiction we have reached proves (2.283).

In view of (2.283), for each $j_1, j_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$d(x_{j_1}, x_{j_2}) \le \epsilon_0 \bar{N}. \tag{2.290}$$

Let

$$k \in \{q\bar{N}, \dots, (q+1)\bar{N}\},\ \ s \in \{1, \dots, m\}. \tag{2.291}$$

By (2.221) and (2.291), there exists

$$j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\} \tag{2.292}$$

such that

$$r(j+1) = s. \tag{2.293}$$

By (2.283), (2.292), and (2.293),

$$d(x_j, P_s(x_j)) = d(x_j, P_{r(j+1)}(x_j)) = d(x_j, x_{j+1}) \le \epsilon_0. \tag{2.294}$$

(A6), (2.255), (2.290)–(2.292), and (2.294) imply that

$$d(x_k, P_s(x_k)) \le d(x_k, x_j) + d(x_j, P_s(x_j)) + d(P_s(x_j), P_s(x_k))$$

$$\le \epsilon_0 + 2d(x_k, x_j) \le \epsilon_0(2\bar{N} + 1) = \epsilon$$

and

$$x_k \in F_\epsilon(P_s) :\ s = 1, \dots, m,\ \ x_k \in F_\epsilon$$

for all integers $k \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$ and all integers $q \ge 2M\gamma^{-1} + 1$. This completes the proof of Theorem 2.11.

2.13 Proof of Theorem 2.12

We may assume that

$$\epsilon_0 < \min\{1, r_0\}/4.$$

We deduce Theorem 2.12 from Theorems 2.9 and 2.11. Let

$$(Y, \rho) = (X, d), \ N = \bar{N}, \ \mathfrak{A} = \{i \to P_{r(i)}, \ i = 1, 2, \dots \ : r \in \mathcal{R}_{per}\},$$

$$F = F_{\epsilon_0/4}. \tag{2.295}$$

In view of Theorem 2.11, for each $M > 0$ there exists an integer $Q > 0$ such that property (P6) holds. Hence Theorem 2.9 implies that there exist $\delta > 0$ and an integer $Q > 0$ such that for each $r \in \mathcal{R}_{per}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies

$$x_0 \in B(\theta, M), \tag{2.296}$$

$$\rho(x_{i+1}, P_{r(i+1)}(x_i)) \le \delta \text{ for all integers } i \ge 0 \tag{2.297}$$

we have

$$B(x_i, \epsilon_0/4) \cap F \neq \emptyset \text{ for all integers } i \ge Q. \tag{2.298}$$

Assume that $r \in \mathcal{R}_{per}$ and that a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies (2.296) and (2.297). Then (2.298) is true.

Assume that an integer $i \ge Q$. By (2.296), there exists

$$\xi \in F \cap B(x_i, \epsilon_0/4). \tag{2.299}$$

In view of (A6), (2.295), and (2.299), for all $s = 1, \dots, m$,

$$d(x_i, P_s(x_i)) \le d(x_i, \xi) + d(\xi, P_s(\xi)) + d(P_s(\xi), P_s(x_i))$$

$$\le 2d(x_i, \xi) + \epsilon_0/4 < \epsilon_0$$

and $x_i \in F_{\epsilon_0}$. Theorem 2.12 is proved.

Chapter 3
Dynamic String-Averaging Methods in Normed Spaces

In this chapter we study the convergence of dynamic string-averaging methods for solving common fixed point problems in a normed space. Our main goal is to obtain an approximate solution of the problem using perturbed algorithms. We show that the inexact dynamic string-averaging algorithm generates an approximate solution if perturbations are summable. We also show that if the mappings are nonexpansive and the perturbations are sufficiently small, then the inexact method produces approximate solutions.

3.1 Preliminaries

Let $(X, \|\cdot\|)$ be a normed space.

For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{\|x - y\| : \ y \in E\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \leq r\}.$$

Let m be a natural number and let $P_i : X \to X$, $i = 1, \ldots, m$ be self-mappings of the space X. Suppose that for every $i \in \{1, \ldots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \neq \emptyset. \tag{3.1}$$

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Set

$$F = \cap_{i=1}^{m} \mathrm{Fix}(P_i). \tag{3.2}$$

Elements of the set F are solutions of common fixed point problem.

For every $\epsilon > 0$ and every $i \in \{1, \dots, m\}$ set

$$F_\epsilon(P_i) = \{x \in X : \|x - P_i(x)\| \le \epsilon\}, \tag{3.3}$$

$$\tilde{F}_\epsilon(P_i) = \{y \in X : d(y, F_\epsilon(P_i)) \le \epsilon\}, \tag{3.4}$$

$$F_\epsilon = \cap_{i=1}^{m} F_\epsilon(P_i), \tag{3.5}$$

$$\tilde{F}_\epsilon = \cap_{i=1}^{m} \tilde{F}_\epsilon(P_i). \tag{3.6}$$

For a given $\epsilon > 0$ a point $x \in \tilde{F}_\epsilon$ is called an ϵ-approximate solution of the common fixed point problem.

We suppose that for every $i \in \{1, \dots, m\}$ the inequality

$$\|z - P_i(x)\| \le \|z - x\| \tag{3.7}$$

holds for all $z \in \mathrm{Fix}(P_i)$ and all $x \in X$.

Suppose that $M_* > 1$ and that the following assumption holds:

(A1) for each $\delta > 0$ there exists $z_\delta \in B(0, M_*)$ such that

$$B(z_\delta, \delta) \cap \mathrm{Fix}(P_i) \ne \emptyset \text{ for all } i = 1, \dots, m.$$

We describe the dynamic string-averaging method with variable strings and weights which is applied in order to obtain a good approximative solution of the common fixed point problem.

By an index vector, we mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q, \; P[t] = P_{t_q} \cdots P_{t_1}. \tag{3.8}$$

It is easy to see that for each index vector t

$$P[t](x) = x \text{ for all } x \in F, \tag{3.9}$$

$$\|P[t](x) - P[t](y)\| = \|x - P[t](y)\| \le \|x - y\| \text{ for all } x \in F \text{ and all } y \in X. \tag{3.10}$$

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{t \in \Omega} w(t) = 1. \tag{3.11}$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{t \in \Omega} w(t) P[t](x), \ \ x \in X. \tag{3.12}$$

It is not difficult to see that

$$P_{\Omega,w}(x) = x \text{ for all } x \in F, \tag{3.13}$$

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| = \|x - P_{\Omega,w}(y)\| \le \|x - y\|$$
$$\text{for all } x \in F \text{ and all } y \in X. \tag{3.14}$$

We use the following dynamic string-averaging algorithm. Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k).$$

Fix a number

$$\Delta \in (0, m^{-1}] \tag{3.15}$$

and an integer

$$\bar{q} \ge m. \tag{3.16}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \le \bar{q} \ \text{ for all } t \in \Omega, \tag{3.17}$$

$$w(t) \ge \Delta \text{ for all } t \in \Omega. \tag{3.18}$$

Fix a natural number $\bar{N}$.

In order to state the main results of this chapter we need the following definitions.

Let $\delta \ge 0$, $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector. Define

$$A_0(x, t, \delta) = \{(y, \lambda) \in X \times R^1 : \text{ there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that}$$
$$y_0 = x \text{ and for all } i = 1, \dots, p(t),$$
$$\|y_i - P_{t_i}(y_{i-1})\| \le \delta,$$
$$y = y_{p(t)},$$
$$\lambda = \max\{\|y_i - y_{i-1}\| : i = 1, \dots, p(t)\}\}. \tag{3.19}$$

Let $\delta \ge 0$, $x \in X$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$A(x, (\Omega, w), \delta) = \{(y, \lambda) \in X \times R^1 : \text{ there exist}$$
$$(y_t, \lambda_t) \in A_0(x, t, \delta),\ t \in \Omega \text{ such that}$$
$$\|y - \sum_{t \in \Omega} w(t) y_t\| \le \delta,\ \lambda = \max\{\lambda_t : t \in \Omega\}\}. \tag{3.20}$$

Denote by $\mathrm{Card}(A)$ the cardinality of a set A. Suppose that the sum over empty set is zero.

3.2 The First Problem

We suppose that $\bar{c} \in (0, 1)$ and that for every $i \in \{1, \dots, m\}$ the inequality

$$\|z - x\|^2 \ge \|z - P_i(x)\|^2 + \bar{c}\|x - P_i(x)\|^2 \tag{3.21}$$

holds for all $x \in X$ and all $z \in \mathrm{Fix}(P_i)$.

In this chapter we prove the following three results: Theorem 3.1 which shows that the inexact dynamic string-averaging method generates approximate solutions if perturbations are summable, Theorem 3.2 which establishes that the exact dynamic string-averaging method generates approximate solutions, and Theorem 3.3 which demonstrates that the inexact dynamic string-averaging method generates approximate solutions if the perturbations are small enough.

Theorem 3.1 *Let*

$$M > M_*, \tag{3.22}$$

$\epsilon \in (0, 1)$ *and let a sequence* $\{\epsilon_i\}_{i=1}^{\infty} \subset (0, \infty)$ *satisfy*

$$\Lambda := \sum_{i=1}^{\infty} \epsilon_i < \infty. \tag{3.23}$$

Let a natural number n_0 *be such that for each integer* $i > n_0$,

$$\epsilon_i < \epsilon(\bar{N}+1)^{-1}(1+\bar{q})^{-1}. \tag{3.24}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{3.25}$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}), \tag{3.26}$$

$$x_0 \in B(0, M) \tag{3.27}$$

and that sequences $\{x_i\}_{i=1}^{\infty} \subset X$, $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *satisfies for each natural number* i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \epsilon_i). \tag{3.28}$$

Then

$$\begin{aligned} &Card(\{i \in \{0, 1, \dots\} : \ x_i \notin \tilde{F}_{\epsilon}\}) \\ &\le n_0 + \bar{N}(1+\bar{N})^2(1+\bar{q})^2\bar{c}^{-1}\Delta^{-1}\epsilon^{-2}((4M + \Lambda(\bar{q}+1))^2 \\ &+\Lambda(2\bar{q}+1)(8M + 2(\bar{q}+1) + \Lambda + 2)). \end{aligned}$$

Theorem 3.1 was obtained in [125].

Theorem 3.2 *Assume that for each* $x, y \in X$ *and each* $i \in \{1, \dots, m\}$,

$$\|P_i(x) - P_i(y)\| \le \|x - y\|. \tag{3.29}$$

Let $M > M_*$, $\epsilon \in (0, 1)$,

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{3.30}$$

satisfy (3.26),

$$(\Omega_{i+\bar{N}}, w_{i+\bar{N}}) = (\Omega_i, w_i) \text{ for all integers } i \ge 0, \tag{3.31}$$

$$x_0 \in B(0, M) \tag{3.32}$$

and let a sequence $\{x_i\}_{i=1}^{\infty} \subset X$ *satisfy for each integer* $i \ge 0$,

$$x_{i+1} = P_{\Omega_{i+1}, w_{i+1}}(x_i). \tag{3.33}$$

Then for each integer $i \geq \bar{N}(1 + 4M^2\bar{c}^{-3}\Delta^{-3}\epsilon^{-4}(8\bar{q}\bar{N})^6)$,

$$x_i \in F_\epsilon.$$

Theorem 3.3 *Assume that for each* $x, y \in X$ *and each* $i \in \{1, \dots, m\}$,

$$\|P_i(x) - P_i(y)\| \leq \|x - y\|.$$

Let $M > M_*$, $r_0 \in (0, 1)$,

$$F_{r_0} \subset B(0, M), \tag{3.34}$$

$$\epsilon_0 \in (0, r_0/2),\ \ Q = \bar{N}(1 + 4^6 M^2\bar{c}^{-3}\Delta^{-3}\epsilon_0^{-4}(8\bar{q}\bar{N})^6), \tag{3.35}$$

$$0 < \delta \leq 4^{-1}\epsilon_0(Q(2\bar{N} + 1))^{-1}(\bar{q} + 1)^{-1}. \tag{3.36}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{3.37}$$

satisfy (3.26),

$$(\Omega_{i+\bar{N}}, w_{i+\bar{N}}) = (\Omega_i, w_i) \textit{ for all integers } i \geq 0, \tag{3.38}$$

$$x_0 \in B(0, M) \tag{3.39}$$

and let sequences $\{x_i\}_{i=1}^{\infty} \subset X$, $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *satisfy for each natural number* i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta). \tag{3.40}$$

Then $x_i \in F_{\epsilon_0}$ *for all integers* $i \geq Q$.

3.3 Proof of Theorem 3.1

In view of assumption (A1), for each positive number δ there is a point

$$z_\delta \in B(0, M_*) \tag{3.41}$$

for which

$$B(z_\delta, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset,\ \ i = 1, \dots, m. \tag{3.42}$$

By (3.42), for every positive number δ and every integer $i \in \{1, \dots, m\}$ there exists a point

$$z_{\delta,i} \in B(z_\delta, \delta) \cap \mathrm{Fixp}(P_i). \tag{3.43}$$

Let a nonnegative integer i be given. In view of (3.28), we have

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), \epsilon_{i+1}). \tag{3.44}$$

It follows from (3.20) and (3.44) that there exist

$$(y_{i,t}, \alpha_{i,t}) \in A_0(x_i, t, \epsilon_{i+1}),\ t \in \Omega_{i+1} \tag{3.45}$$

such that

$$\|x_{i+1} - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\| \le \epsilon_{i+1}, \tag{3.46}$$

$$\lambda_{i+1} = \max\{\alpha_{i,t} :\ t \in \Omega_{i+1}\}. \tag{3.47}$$

By (3.19) and (3.45), for each index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(i,t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(i,t)} = x_i,\ y_{p(t)}^{(i,t)} = y_{i,t}, \tag{3.48}$$

for every integer $j = 1, \dots, p(t)$, we have

$$\|y_j^{(i,t)} - P_{t_j}(y_{j-1}^{(i,t)})\| \le \epsilon_{i+1}, \tag{3.49}$$

$$\alpha_{i,t} = \max\{\|y_{j+1}^{(i,t)} - y_j^{(i,t)}\| :\ j = 0, \dots, p(t) - 1\}. \tag{3.50}$$

Set

$$\epsilon_0 = 0. \tag{3.51}$$

In view of (3.27) and (3.41), we have

$$\|z_\delta - x_0\| \le 2M. \tag{3.52}$$

Let a nonnegative integer i be given and

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1},\ j = 0, \dots, p(t) - 1. \tag{3.53}$$

It follows from (3.7), (3.43), and (3.49) that

$$\|z_\delta - y_{j+1}^{(i,t)}\| \le \|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\| + \|P_{t_{j+1}}(y_j^{(i,t)}) - y_{j+1}^{(i,t)}\|$$
$$\le \|z_\delta - z_{\delta,t_{j+1}}\| + \|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\| + \epsilon_{i+1}$$
$$\le \delta + \epsilon_{i+1} + \|z_{\delta,t_{j+1}} - y_j^{(i,t)}\|$$
$$\le \|z_\delta - y_j^{(i,t)}\| + 2\delta + \epsilon_{i+1}$$

and

$$\|z_\delta - y_{j+1}^{(i,t)}\| \le \|z_\delta - y_j^{(i,t)}\| + 2\delta + \epsilon_{i+1}. \tag{3.54}$$

It is clear that

$$\|z_\delta - y_j^{(i,t)}\|^2 - \|z_\delta - y_{j+1}^{(i,t)}\|^2$$
$$\ge \|z_\delta - y_j^{(i,t)}\|^2 - \|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\|^2$$
$$+\|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\|^2 - \|z_\delta - y_{j+1}^{(i,t)}\|^2$$
$$\ge \|z_\delta - y_j^{(i,t)}\|^2 - \|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\|^2$$
$$-\|y_{j+1}^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|(\|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\| + \|z_\delta - y_{j+1}^{(i,t)}\|). \tag{3.55}$$

Relations (3.21), (3.42), and (3.43) imply that

$$\|z_\delta - y_j^{(i,t)}\|^2 - \|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\|^2$$
$$\ge \|z_{\delta,t_{j+1}} - y_j^{(i,t)}\|^2 - \|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\|^2$$
$$-\|z_{\delta,t_{j+1}} - y_j^{(i,t)}\|^2 + \|z_\delta - y_j^{(i,t)}\|^2$$
$$+\|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\|^2 - \|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\|^2$$
$$\ge \bar{c}\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|^2$$
$$-\|z_\delta - z_{\delta,t_{j+1}}\|(\|z_{\delta,t_{j+1}} - y_j^{(i,t)}\| + \|z_\delta - y_j^{(i,t)}\|)$$
$$-\|z_\delta - z_{\delta,t_{j+1}}\|(\|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\| + \|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\|). \tag{3.56}$$

In view of (3.7), (3.42), (3.43), and (3.56),

$$\|z_\delta - y_j^{(i,t)}\|^2 - \|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\|^2$$
$$\ge \bar{c}\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|^2$$

$$
\begin{aligned}
&-\delta(\|z_{\delta,t_{j+1}} - y_j^{(i,t)}\| + \|z_\delta - y_j^{(i,t)}\|)\\
&-\delta(\|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\| + \|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\|)\\
&\ge \bar{c}\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|^2 - \delta(2\|z_\delta - y_j^{(i,t)}\| + \delta)\\
&-\delta(2\|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\| + \delta)\\
&\ge \bar{c}\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|^2 - \delta(2\|z_\delta - y_j^{(i,t)}\| + \delta)\\
&-\delta(2\|z_{\delta,t_{j+1}} - y_j^{(i,t)}\| + \delta)\\
&\ge \bar{c}\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|^2 - \delta(2\|z_\delta - y_j^{(i,t)}\| + \delta)\\
&-\delta(2\|z_\delta - y_j^{(i,t)}\| + 3\delta)\\
&\ge \bar{c}\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|^2 - 2\delta(2\|z_\delta - y_j^{(i,t)}\| + 3\delta).
\end{aligned}
\tag{3.57}
$$

By (3.49), (3.55), and (3.57), we have

$$
\begin{aligned}
&\|z_\delta - y_j^{(i,t)}\|^2 - \|z_\delta - y_{j+1}^{(i,t)}\|^2\\
&\ge \bar{c}\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|^2 - 2\delta(2\|z_\delta - y_j^{(i,t)}\| + 3\delta)\\
&-\|y_{j+1}^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|(\|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\| + \|z_\delta - y_{j+1}^{(i,t)}\|)\\
&\ge \bar{c}\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|^2 - 2\delta(2\|z_\delta - y_j^{(i,t)}\| + 3\delta)\\
&-\epsilon_{i+1}(\|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\| + \|z_\delta - y_{j+1}^{(i,t)}\|).
\end{aligned}
\tag{3.58}
$$

It follows from (3.7) and (3.43) that

$$
\begin{aligned}
\|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\| &\le \|z_\delta - z_{\delta,t_{j+1}}\| + \|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\|\\
&\le \delta + \|z_{\delta,t_{j+1}} - y_j^{(i,t)}\|\\
&\le \delta + \|z_{\delta,t_{j+1}} - z_\delta\| + \|z_\delta - y_j^{(i,t)}\|\\
&\le 2\delta + \|z_\delta - y_j^{(i,t)}\|.
\end{aligned}
\tag{3.59}
$$

By (3.49), (3.58), and (3.59), we have

$$
\begin{aligned}
&\|z_\delta - y_j^{(i,t)}\|^2 - \|z_\delta - y_{j+1}^{(i,t)}\|^2\\
&\ge \bar{c}\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|^2 - 2\delta(2\|z_\delta - y_j^{(i,t)}\| + 3\delta)\\
&-\epsilon_{i+1}(2\|z_\delta - y_j^{(i,t)}\| + 4\delta + \epsilon_{i+1}).
\end{aligned}
\tag{3.60}
$$

Relations (3.17), (3.48), and (3.54) imply that for all integers $j = 0, \dots, p(t)$,

$$\begin{aligned}\|z_\delta - y_j^{(i,t)}\| &\le \|z_\delta - y_0^{(i,t)}\| + j(2\delta + \epsilon_{i+1}) \\ &= \|z_\delta - x_i\| + j(2\delta + \epsilon_{i+1}) \\ &\le \|z_\delta - x_i\| + p(t)(2\delta + \epsilon_{i+1}) \\ &\le \|z_\delta - x_i\| + \bar{q}(2\delta + \epsilon_{i+1}). \end{aligned} \tag{3.61}$$

In view of (3.48) and (3.61),

$$\|z_\delta - y_{i,t}\| = \|z_\delta - y_{p(t)}^{(i,t)}\| \le \|z_\delta - x_i\| + q(2\delta + \epsilon_{i+1}). \tag{3.62}$$

It follows from (3.11), (3.46), (3.62), and the convexity of the norm that

$$\begin{aligned}\|z_\delta - x_{i+1}\| &\le \|z_\delta - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\| + \|\sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t} - x_{i+1}\| \\ &\le \sum_{t \in \Omega_{i+1}} w_{i+1}(t) \|z_\delta - y_{i,t}\| + \epsilon_{i+1} \\ &\le \|z_\delta - x_i\| + (\bar{q} + 1)(2\delta + \epsilon_{i+1}). \end{aligned} \tag{3.63}$$

By induction we show that for all integers $i \ge 0$,

$$\|z_\delta - x_i\| \le 2M + 2(\bar{q} + 1)\delta i + (\sum_{j=0}^{i} \epsilon_j)(\bar{q} + 1). \tag{3.64}$$

By (3.51) and (3.52), inequality (3.64) holds for $i = 0$.

Assume that i is a nonnegative integer and that (3.64) is true. In view of (3.63) and (3.64),

$$\begin{aligned}\|z_\delta - x_{i+1}\| &\le \|z_\delta - x_i\| + (\bar{q} + 1)(2\delta + \epsilon_{i+1}) \\ &\le 2M + 2(\bar{q} + 1)\delta(i + 1) + (\sum_{j=0}^{i+1} \epsilon_j)(\bar{q} + 1). \end{aligned}$$

Therefore by induction we showed that inequality (3.64) is true for all nonnegative integers i.

Relations (3.61) and (3.64) imply that for all nonnegative integers i, all index vectors $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$, and all integers $j = 0, \dots, p(t)$,

$$\begin{aligned}\|z_\delta - y_j^{(i,t)}\| &\le \|z_\delta - x_i\| + \bar{q}(2\delta + \epsilon_{i+1}) \\ &\le 2M + 2(\bar{q} + 1)\delta(i + 1) + (\sum_{j=0}^{i+1} \epsilon_j)(\bar{q} + 1). \end{aligned} \tag{3.65}$$

Let a natural number n be given. In view of (3.23), (3.41), (3.64), and (3.65), for all integers $i = 0, \dots, n$, all index vectors $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$, and all integers $j = 0, \dots, p(t)$,

$$\|x_i\| \le \|z_\delta\| + \|x_i - z_\delta\| \le 3M + 2(\bar{q}+1)\delta n + \Lambda(\bar{q}+1),$$

$$\|y_j^{(i,t)}\| \le \|z_\delta\| + \|y_j^{(i,t)} - z_\delta\| \le 3M + 2(\bar{q}+1)\delta n + \Lambda(\bar{q}+1).$$

Since the relation above holds for every number $\delta \in (0, 1)$ we conclude that for all nonnegative integers i, all index vectors $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all integers $j = 0, \dots, p(t)$,

$$\|x_i\| \le 3M + \Lambda(\bar{q}+1), \tag{3.66}$$

$$\|y_j^{(i,t)}\| \le 3M + +\Lambda(\bar{q}+1). \tag{3.67}$$

It follows from (3.41), (3.60), and (3.67) that for each positive number δ, each nonnegative integer i, each index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$, and each integer $j = 0, \dots, p(t) - 1$,

$$\begin{aligned}
&\|z_\delta - y_j^{(i,t)}\|^2 - \|z_\delta - y_{j+1}^{(i,t)}\|^2 \\
&\ge \bar{c}\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|^2 \\
&-2\delta(2(4M + (\bar{q}+1)\Lambda) + 3\delta) \\
&= \epsilon_{i+1}(2(4M + (\bar{q}+1)\Lambda) + 3\delta + \epsilon_{i+1}).
\end{aligned} \tag{3.68}$$

By (3.49), for each positive number δ, each nonnegative integer i, each index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$, and each integer $j = 0, \dots, p(t) - 1$,

$$\begin{aligned}
&\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|^2 \ge \|y_j^{(i,t)} - y_{j+1}^{(i,t)}\|^2 \\
&-(\|y_j^{(i,t)} - y_{j+1}^{(i,t)}\|^2 - \|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|^2) \\
&\ge \|y_j^{(i,t)} - y_{j+1}^{(i,t)}\|^2 \\
&-\|y_{j+1}^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|(\|y_j^{(i,t)} - y_{j+1}^{(i,t)}\| + \|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|) \\
&\ge \|y_j^{(i,t)} - y_{j+1}^{(i,t)}\|^2 \\
&-\epsilon_{i+1}(2\|y_j^{(i,t)} - y_{j+1}^{(i,t)}\| + \|y_{j+1}^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|) \\
&\ge \|y_j^{(i,t)} - y_{j+1}^{(i,t)}\|^2 - \epsilon_{i+1}(2(3M + (\bar{q}+1)\Lambda) + \epsilon_{i+1}).
\end{aligned} \tag{3.69}$$

Relations (3.68) and (3.69) imply that for every positive number δ, every nonnegative integer i, every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$, and every integer $j = 0, \dots, p(t) - 1$, we have

$$\|z_\delta - y_j^{(i,t)}\|^2 - \|z_\delta - y_{j+1}^{(i,t)}\|^2$$
$$\geq \bar{c}\|y_j^{(i,t)} - y_{j+1}^{(i,t)}\|^2 - 2\delta(2(4M + (\bar{q}+1)\Lambda) + 3\delta)$$
$$-\epsilon_{i+1}(2(7M + 2(\bar{q}+1)\Lambda) + 3\delta + 2\epsilon_{i+1}). \tag{3.70}$$

Let a nonnegative integer i be given and let $\delta \in (0,1)$. It follows from (3.3), (3.17), (3.48), (3.50), and (3,70) that for all index vectors $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$,

$$\|z_\delta - x_i\|^2 - \|z_\delta - y_{i,t}\|^2$$
$$= \|z_\delta - y_0^{(i,t)}\|^2 - \|z_\delta - y_{p(t)}^{(i,t)}\|^2$$
$$= \sum_{j=0}^{p(t)-1} [\|z_\delta - y_j^{(i,t)}\|^2 - \|z_\delta - y_{j+1}^{(i,t)}\|^2]$$
$$\geq \bar{c} \sum_{j=0}^{p(t)-1} \|y_j^{(i,t)} - y_{j+1}^{(i,t)}\|^2$$
$$-2\delta\bar{q}(2(4M + (\bar{q}+1)\Lambda) + 3\delta)$$
$$-\epsilon_{i+1}\bar{q}(2(7M + 2(\bar{q}+1)\Lambda) + 3\delta + 2\epsilon_{i+1})$$
$$\geq \bar{c}\alpha_{i,t}^2 - 2\delta\bar{q}(2(4M + (\bar{q}+1)\Lambda) + 3\delta)$$
$$-\epsilon_{i+1}\bar{q}(2(8M + 2(\bar{q}+1)\Lambda) + 3\delta + 2\epsilon_{i+1}). \tag{3.71}$$

By (3.11), (3.18), (3.41), (3.46), (3.47), (3.66), (3.67), (3.71), and the convexity of the function $\|\cdot\|^2$, we have

$$\|z_\delta - x_{i+1}\|^2$$
$$= \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t}\|^2 + \|z_\delta - x_{i+1}\|^2$$
$$-\|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t}\|^2$$
$$\leq \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t}\|^2$$
$$+(\|x_{i+1} - z_\delta\| - \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t}\|)$$
$$\times(\|x_{i+1} - z_\delta\| + \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t}\|)$$

$$\leq \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)(t)y_{i,t}\|^2 + 2\epsilon_{i+1}(4M + \Lambda(\bar{q}+1))$$

$$\leq \sum_{t\in\Omega_{i+1}} w_{i+1}(t)\|z_\delta - y_{i,t}\|^2 + 2\epsilon_{i+1}(4M + \Lambda(\bar{q}+1))$$

$$\leq \sum_{t\in\Omega_{i+1}} w_{i+1}(t)[\|z_\delta - x_i\|^2 - \bar{c}\alpha_{i,t}^2$$

$$+2\delta\bar{q}(8M + 2(\bar{q}+1)\Lambda + 3\delta)$$

$$+\epsilon_{i+1}\bar{q}(2(8M + 2(\bar{q}+1)\Lambda) + 3\delta + 2\epsilon_{i+1})]$$

$$+2\epsilon_{i+1}(4M + \Lambda(\bar{q}+1))$$

$$\leq \|z_\delta - x_i\|^2 - \bar{c}\Delta\lambda_{i+1}^2$$

$$+\epsilon_{i+1}\bar{q}(2(8M + 2(\bar{q}+1)\Lambda + 3\delta + 2\epsilon_{i+1}))$$

$$+2\delta\bar{q}(8M + 2(\bar{q}+1)\Lambda + 3\delta) + 2\epsilon_{i+1}(4M + \Lambda(\bar{q}+1)). \tag{3.72}$$

Define

$$\gamma_0 = \epsilon(\bar{N}+1)^{-1}(1+\bar{q})^{-1}. \tag{3.73}$$

By (3.24) and (3.73), for every integer $i > n_0$, we have

$$\epsilon_i < \gamma_0. \tag{3.74}$$

Relations (3.41), (3.66), and (3.72) imply that for every integer $n > n_0$,

$$(4M + (\bar{q}+1)\Lambda)^2 \geq \|z_\delta - x_{n_0}\|^2$$

$$\geq \|z_\delta - x_{n_0}\|^2 - \|z_\delta - x_n\|^2$$

$$= \sum_{i=n_0}^{n-1}(\|z_\delta - x_i\|^2 - \|z_\delta - x_{i+1}\|^2)$$

$$\geq \sum_{i=n_0}^{n-1} \bar{c}\Delta\lambda_{i+1}^2 - \sum_{i=n_0}^{n-1} \epsilon_{i+1}\bar{q}(2(8M + 2(\bar{q}+1)\Lambda + 3\delta + 2\epsilon_{i+1}))$$

$$-2(n-n_0)\delta\bar{q}(8M + 2(\bar{q}+1)\Lambda + 3\delta) - 2\sum_{i=n_0}^{n-1}\epsilon_{i+1}(4M + \Lambda(\bar{q}+1)).$$

Since δ is any element of the interval $(0, 1)$, it follows from (3.23) and (3.74) that for every integer $n > n_0$,

$$(4M + (\bar{q} + 1)\Lambda)^2 + (2\bar{q}(8M + 2(\bar{q} + 1)\Lambda + 2) + 8M + 2\Lambda(\bar{q} + 1))\Lambda$$
$$\geq \sum_{i=n_0}^{n-1} \bar{c}\Delta\lambda_{i+1}^2$$
$$\geq \bar{c}\Delta\gamma_0^2 \mathrm{Card}(\{k \in \{n_0, \dots, n-1\} : \ \lambda_{k+1} \geq \gamma_0\}).$$

Since the relation above is true for every natural number $n > n_0$ we conclude that

$$\mathrm{Card}(\{k \in \{n_0, n_0 + 1, \dots, \} : \ \lambda_{k+1} \geq \gamma_0\})$$
$$\leq \bar{c}^{-1}\Delta^{-1}\gamma_0^{-2}[(4M + (\bar{q} + 1)\Lambda)^2$$
$$+\Lambda(2\bar{q} + 1)(8M + 2(\bar{q} + 1)\Lambda + 2)]. \quad (3.75)$$

Assume that an integer $i > 0$ satisfies

$$i \geq n_0, \ \lambda_{i+1} < \gamma_0. \quad (3.76)$$

Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ be given. In view of (3.47), (3.49), (3.50), and (3.76), for all integers $j = 0, \dots, p(t) - 1$, we have

$$\gamma_0 > \|y_{j+1}^{(i,t)} - y_j^{(i,t)}\|$$
$$\geq \|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\| - \|y_{j+1}^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|$$
$$\geq \|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\| - \epsilon_{i+1}. \quad (3.77)$$

By (3.3), (3.74), (3.76), and (3.77), for all integers $j = 0, \dots, p(t) - 1$,

$$\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\| < 2\gamma_0 \quad (3.78)$$

and

$$y_j^{(i,t)} \in F_{2\gamma_0}(P_{t_{j+1}}). \quad (3.79)$$

It follows from (3.17), (3.48), (3.77), and (3.78) that for all integers $j = 0, \dots, p(t)$,

$$\|x_i - y_j^{(i,t)}\| \leq j\gamma_0 \leq \bar{q}\gamma_0 \quad (3.80)$$

and if $j < p(t)$, then

$$x_i \in \tilde{F}_{(\bar{q}+1)\gamma_0}(P_{t_{j+1}}).$$

Thus

$$x_i \in \tilde{F}_{(\bar{q}+1)\gamma_0}(P_{t_s}), \ s = 1, \dots, p(t). \tag{3.81}$$

In view of (3.48) and (3.80),

$$\|x_i - y_{i,t}\| \le \bar{q}\gamma_0 \text{ for all } t \in \Omega_{i+1}. \tag{3.82}$$

By (3.81), we have

$$x_i \in \cap\{\tilde{F}_{(\bar{q}+1)\gamma_0}(P_s) : \ s \in \cup_{t\in\Omega_{i+1}}\{t_1, \dots, t_{p(t)}\}\}. \tag{3.83}$$

In view of (3.11), (3.46), (3.74), (3.76), (3.82), and the convexity of the norm,

$$\begin{aligned}
&\|x_{i+1} - x_i\| \\
&\le \|x_{i+1} - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t}\| + \|\sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t} - x_i\| \\
&\le \epsilon_{i+1} + \|\sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t} - x_i\| \\
&\le \epsilon_{i+1} + \gamma_0\bar{q} < \gamma_0(\bar{q}+1),
\end{aligned}$$

$$\|x_{i+1} - x_i\| < \gamma_0(\bar{q}+1). \tag{3.84}$$

Define

$$E_0 = \{i \in \{n_0, n_0+1, \dots\} : \ \lambda_{i+1} \ge \gamma_0\}. \tag{3.85}$$

By (3.75) and (3.83)–(3.85),

$$\mathrm{Card}(E_0) \le \bar{c}^{-1}\Delta^{-1}\gamma_0^{-2}[(4M + (\bar{q}+1)\Lambda)^2 + \Lambda(2\bar{q}+1)(8M + 2(\bar{q}+1)\Lambda + 2)] \tag{3.86}$$

and the following property holds:

(P1) if a natural number $i \ge n_0$ satisfies $\lambda_{i+1} < \gamma_0$, then

$$x_i \in \tilde{F}_{(\bar{q}+1)\gamma_0}(P_s), \ s \in \cup_{t\in\Omega_{i+1}}\{t_1, \dots, t_{p(t)}\}, \tag{3.87}$$

$$\|x_{i+1} - x_i\| < \gamma_0(\bar{q}+1). \tag{3.88}$$

Define

$$E_1 = \{i \in \{n_0, n_0+1, \dots\} : \ [i, i+\bar{N}-1] \cap E_0 \ne \emptyset\}. \tag{3.89}$$

It follows from (3.86) and (3.89) that

$$\mathrm{Card}(E_1) \le \bar{N}\mathrm{Card}(E_0)$$
$$\le \bar{N}\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2}[(4M + (\bar{q}+1)\Lambda)^2 + \Lambda(2\bar{q}+1)(8M + 2(\bar{q}+1) + \Lambda + 2)] \tag{3.90}$$

Let an integer $j \ge n_0$ satisfy

$$j \notin E_1. \tag{3.91}$$

By (3.91), we have

$$[j, j + \bar{N} - 1] \cap E_0 = \emptyset. \tag{3.92}$$

It follows from property (P1), (3.85), and (3.92) that for every integer $i \in \{j, \dots, j + \bar{N} - 1\}$, $\lambda_{i+1} < \gamma_0$ and (3.87) and (3.88) are true. By (3.88) which holds for every integer $i \in \{j, \dots, j + \bar{N} - 1\}$, for every pair of integers $i_1, i_2 \in \{j, \dots, j + \bar{N}\}$, we have

$$\|x_{i_1} - x_{i_2}\| \le (\bar{q}+1)\bar{N}\gamma_0. \tag{3.93}$$

In view of (3.87) which is valid for every integer $i \in \{j, \dots, j + \bar{N} - 1\}$, (3.26) and (3.93),

$$x_j \in \tilde{F}_{(\bar{q}+1)\gamma_0(\bar{N}+1)}(P_s),$$
$$s \in \cup_{i=j}^{j+\bar{N}-1} \cup \{\{t_1, \dots, t_{p(t)}\} : t = \{t_1, \dots, t_{p(t)}\} \in \Omega_{i+1}\} = \{1, \dots, m\}.$$

By the relation above and (3.73),

$$x_j \in \tilde{F}_{(\bar{q}+1)\gamma_0(\bar{N}+1)} = \tilde{F}_\epsilon$$

for all integers $j \ge n_0$ such that $j \notin E_1$. Combined with (3.73) and (3.90) this implies that

$$\mathrm{Card}(\{j \in \{0, 1, \dots\} : \ x_j \notin \tilde{F}_\epsilon\})$$
$$\le n_0 + \mathrm{Card}(E_1)$$
$$\le n_0 + \bar{N}(1+\bar{N})^2(1+\bar{q})^2\bar{c}^{-1}\Delta^{-1}\epsilon^{-2}((4M + \Lambda(\bar{q}+1))^2$$
$$+\Lambda(2\bar{q}+1)(8M + 2(\bar{q}+1) + \Lambda + 2)).$$

Theorem 3.1 is proved.

3.4 Proof of Theorem 3.2

Set

$$\gamma_0 = \epsilon^2 (8\bar{q}\bar{N})^{-3}\bar{c}\Delta \tag{3.94}$$

and

$$T = \prod_{i=1}^{\bar{N}} P_{\Omega_i, w_i} = P_{\Omega_{\bar{N}}, w_{\bar{N}}} \cdots P_{\Omega_1, w_1}. \tag{3.95}$$

By (3.8), (3.12), and (3.29), for each integer $i \geq 0$, each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and all $x, y \in X$,

$$\|P[t](x) - P[t](y)\| \leq \|x - y\|, \tag{3.96}$$

$$\|P_{\Omega_{i+1}, w_{i+1}}(x) - P_{\Omega_{i+1}, w_{i+1}}(y)\|$$
$$= \|\sum_{t \in \Omega_{i+1}} w_{i+1}(t) P[t](x) - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) P[t](y)\| \leq \|x - y\|. \tag{3.97}$$

In view of (3.31) and (3.33), for each integer $i \geq 0$,

$$x_{(i+1)\bar{N}} = P_{\Omega_{(i+1)\bar{N}}, w_{(i+1)\bar{N}}} \cdots P_{\Omega_{i\bar{N}+1}, w_{i\bar{N}+1}}(x_{i\bar{N}}) = T(x_{i\bar{N}}). \tag{3.98}$$

By (A1), for every $\delta > 0$ there exists

$$z_\delta \in B(0, M_*) \tag{3.99}$$

such that

$$B(z_\delta, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset, \ \ i = 1, \ldots, m. \tag{3.100}$$

In view of (3.100), for each $\delta > 0$ and each $i \in \{1, \ldots, m\}$ there exists

$$z_{\delta,i} \in B(z_\delta, \delta) \cap \mathrm{Fixp}(P_i). \tag{3.101}$$

Let $i \geq 0$ be an integer. By (3.19), (3.20), and (3.33), there exists $\lambda_{i+1} \geq 0$ such that

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), 0). \tag{3.102}$$

By (3.20) and (3.102), there exist vectors

$$(y_{i,t}, \alpha_{i,t}) \in A_0(x_i, t, 0), \; t \in \Omega_{i+1} \tag{3.103}$$

such that

$$x_{i+1} = \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}, \tag{3.104}$$

$$\lambda_{i+1} = \max\{\alpha_{i,t} : \; t \in \Omega_{i+1}\}. \tag{3.105}$$

By (3.19) and (3.103), for every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(i,t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(i,t)} = x_i, \; y_{p(t)}^{(i,t)} = y_{i,t} \tag{3.106}$$

for every integer $j = 1, \dots, p(t)$,

$$y_j^{(i,t)} = P_{t_j}(y_{j-1}^{(i,t)}), \tag{3.107}$$

$$\alpha_{i,t} = \max\{\|y_{j+1}^{(i,t)} - y_j^{(i,t)}\| : \; j = 0, \dots, p(t) - 1\}. \tag{3.108}$$

Let $\delta > 0$. By (3.32) and (3.100),

$$\|z_\delta - x_0\| \le 2M. \tag{3.109}$$

Let $i \ge 0$ be an integer, $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$, $j = 0, \dots, p(t) - 1$. In view of (3.7), (3.101), and (3.107),

$$\begin{aligned} \|z_\delta - y_{j+1}^{(i,t)}\| &\le \|z_\delta - z_{\delta,t_{j+1}}\| + \|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\| \\ &\le \delta + \|z_{\delta,t_{j+1}} - y_j^{(i,t)}\| \\ &\le 2\delta + \|z_\delta - y_j^{(i,t)}\|. \end{aligned} \tag{3.110}$$

It follows from (3.17), (3.106), and (3.110) that for all $j = 0, \dots, p(t)$,

$$\begin{aligned} &\|z_\delta - y_j^{(i,t)}\| \le \|z_\delta - y_0^{(i,t)}\| + 2\delta j \\ &= \|z_\delta - x_i\| + 2\delta j \le \|z_\delta - x_i\| + 2\delta \bar{q}. \end{aligned} \tag{3.111}$$

In view of (3.106) and (3.111),

$$\begin{aligned} \|z_\delta - y_{i,t}\| &= \|z_\delta - y_{p(t)}^{(i,t)}\| \\ &\le \|z_\delta - x_i\| + 2\delta \bar{q}. \end{aligned} \tag{3.112}$$

By (3.11), (3.104), (3.112), and the convexity of the norm,

$$\|z_\delta - x_{i+1}\| = \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t}\|$$
$$\le \sum_{t\in\Omega_{i+1}} w_{i+1}(t)\|z_\delta - y_{i,t}\|$$
$$\le \|z_\delta - x_i\| + 2\bar{q}\delta. \tag{3.113}$$

By (3.109) and (3.113), for all integers $i \ge 0$,

$$\|z_\delta - x_i\| \le \|z_\delta - x_0\| + 2\delta\bar{q}i \le 2M + 2\delta\bar{q}i. \tag{3.114}$$

It follows from (3.109), (3.111), and (3.114) that for all integers $i \ge 0$, all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \ldots, p(t)$,

$$\|z_\delta - y_j^{(i,t)}\| \le \|z_\delta - x_i\| + 2\delta\bar{q} \le 2M + 2\delta\bar{q}(i+1). \tag{3.115}$$

By (3.100), (3.114), and (3.115), for all integers $i \ge 0$, all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \ldots, p(t)$,

$$\|x_i\| \le \|z_\delta\| + \|x_i - z_\delta\| \le 3M + 2\bar{q}\delta i,$$
$$\|y_j^{(i,t)}\| \le \|z_\delta\| + \|y_j^{(i,t)} - z_\delta\| \le 3M + 2\bar{q}\delta(i+1).$$

Since the relation above holds for any $\delta > 0$ we conclude that for all integers $i \ge 0$, all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \ldots, p(t)$,

$$\|x_i\| \le 3M,\ \|y_j^{(i,t)}\| \le 3M. \tag{3.116}$$

Let $\delta \in (0, 1)$, an integer $i \ge 0$, $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and $j \in \{0, \ldots, p(t) - 1\}$. By (3.99), (3.101), (3.107), and (3.116),

$$\|z_\delta - y_j^{(i,t)}\|^2 - \|z_\delta - y_{j+1}^{(i,t)}\|^2$$
$$\ge \|z_{\delta,t_{j+1}} - y_j^{(i,t)}\|^2 - \|z_{\delta,t_{j+1}} - y_{j+1}^{(i,t)}\|^2$$
$$-(\|z_{\delta,t_{j+1}} - y_j^{(i,t)}\|^2 - \|z_\delta - y_j^{(i,t)}\|^2)$$
$$+\|z_{\delta,t_{j+1}} - y_{j+1}^{(i,t)}\|^2 - \|z_\delta - y_{j+1}^{(i,t)}\|^2$$
$$\ge \|z_{\delta,t_{j+1}} - y_j^{(i,t)}\|^2 - \|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\|^2$$
$$-\|z_{\delta,t_{j+1}} - z_\delta\|(\|z_\delta - y_j^{(i,t)}\| + \|z_{\delta,t_{j+1}} - y_j^{(i,t)}\|)$$

$$-\|z_{\delta,t_{j+1}} - z_\delta\|(\|z_\delta - y_j^{(i,t)}\| + \|z_{\delta,t_{j+1}} - y_{j+1}^{(i,t)}\|)$$
$$\geq \|z_{\delta,t_{j+1}} - y_j^{(i,t)}\|^2 - \|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\|^2 - 2\delta(8M+1)$$
$$\geq \bar{c}\|y_j^{(i,t)} - y_{j+1}^{(i,t)}\|^2 - 2\delta(8M+1). \tag{3.117}$$

By (3.17), (3.106), (3.108), and (3.117),

$$\|z_\delta - x_i\|^2 - \|z_\delta - y_{i,t}\|^2$$
$$= \|z_\delta - y_0^{(i,t)}\|^2 - \|z_\delta - y_{p(t)}^{(i,t)}\|^2$$
$$= \sum_{j=0}^{p(t)-1} [\|z_\delta - y_j^{(i,t)}\|^2 - \|z_\delta - y_{j+1}^{(i,t)}\|^2]$$
$$\geq \sum_{j=0}^{p(t)-1} \bar{c}\|y_j^{(i,t)} - y_{j+1}^{(i,t)}\|^2 - 2\delta\bar{q}(8M+1)$$
$$\geq \bar{c}\alpha_{i,t}^2 - 2\delta\bar{q}(8M+1). \tag{3.118}$$

It follows from (3.11), (3.18), (3.104), (3.105), (3.118), and the convexity of the function $\|\cdot\|^2$ that

$$\|z_\delta - x_{i+1}\|^2$$
$$= \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t}\|^2$$
$$\leq \sum_{t\in\Omega_{i+1}} w_{i+1}(t)\|z_\delta - y_{i,t}\|^2$$
$$\leq \|z_\delta - x_i\|^2 - \bar{c}\sum_{t\in\Omega_{i+1}} w_{i+1}(t)\alpha_{i,t}^2 + 2\delta\bar{q}(8M+1)$$
$$\leq \|z_\delta - x_i\|^2 - \bar{c}\Delta\lambda_{i+1}^2 + 2\delta\bar{q}(8M+1). \tag{3.119}$$

Let n be a natural number. By (3.109) and (3.119),

$$4M^2 \geq \|z_\delta - x_0\|^2$$
$$\geq \|z_\delta - x_0\|^2 - \|z_\delta - x_{\bar{N}n}\|^2$$
$$= \sum_{k=0}^{n-1}(\|z_\delta - x_{k\bar{N}}\|^2 - \|z_\delta - x_{(k+1)\bar{N}}\|^2)$$

$$= \sum_{k=0}^{n-1} (\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (\|z_\delta - x_j\|^2 - \|z_\delta - x_{j+1}\|^2))$$

$$\geq \sum_{k=0}^{n-1} \sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (\bar{c}\Delta\lambda_{i+1}^2 + 2\delta\bar{q}(8M+1)).$$

Since δ is any element of the interval (0, 1) we conclude that

$$4M^2 \geq \sum_{k=0}^{n-1} \sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (\bar{c}\Delta\lambda_{i+1}^2)$$

$$\geq \bar{c}\Delta\gamma_0^2 \mathrm{Card}(\{k \in \{0, \dots, n-1\} :$$

$$\max\{\lambda_{i+1} : \ i = k\bar{N}, \dots, (k+1)\bar{N} - 1\} \geq \gamma_0\})$$

and

$$\mathrm{Card}(\{k \in \{0, \dots, n-1\} :$$

$$\max\{\lambda_{i+1} : \ i = k\bar{N}, \dots, (k+1)\bar{N} - 1\} \geq \gamma_0\})$$

$$\leq 4M^2\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2}.$$

Since the relation above holds for every natural number n we conclude that

$$\mathrm{Card}(\{k \in \{0, 1, \dots\} : \ \max\{\lambda_{i+1} : \ i = k\bar{N}, \dots, (k+1)\bar{N} - 1\} \geq \gamma_0\})$$
$$\leq 4M^2\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2}. \tag{3.120}$$

In view of (3.120), there exists an integer $q_0 \geq 0$ such that

$$q_0 \leq 4M^2\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2} + 1 \tag{3.121}$$

and

$$\lambda_{i+1} < \gamma_0, \ i = q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1. \tag{3.122}$$

By (3.17), (3.105), (3.106), (3.108), and (3.122), for all integers

$$i = q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1,$$

all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t) - 1$,

$$\|y_j^{(i,t)} - y_{j+1}^{(i,t)}\| \le \gamma_0,$$

$$\|x_i - y_{i,t}\| = \|y_0^{(i,t)} - y_{p(t)}^{(i,t)}\|$$
$$\le \sum_{j=0}^{p(t)-1} \|y_j^{(i,t)} - y_{j+1}^{(i,t)}\| \le \bar{q}\gamma_0. \tag{3.123}$$

It follows from (3.11), (3.104), and the convexity of the function $\|\cdot\|$ that for all $i = q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1$,

$$\|x_i - x_{i+1}\|$$
$$\le \|x_i - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\|$$
$$\le \sum_{t \in \Omega_{i+1}} w_{i+1}(t)\|x_i - y_{i,t}\| \le \bar{q}\gamma_0. \tag{3.124}$$

In view of (3.124),

$$\|x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}}\| \le \bar{q}\gamma_0\bar{N}. \tag{3.125}$$

By (3.95), (3.97), (3.98), and (3.125), for each integer $q > q_0$,

$$\|x_{q\bar{N}} - x_{(q+1)\bar{N}}\| = \|T^{q-q_0}(x_{q_0\bar{N}}) - T^{q-q_0}(x_{(q_0+1)\bar{N}})\|$$
$$\le \|x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}}\| \le \bar{q}\gamma_0\bar{N}. \tag{3.126}$$

Let $q \ge q_0$ be an integer. In view of (3.125) and (3.126),

$$\|x_{q\bar{N}} - x_{(q+1)\bar{N}}\| \le \bar{q}\gamma_0\bar{N}. \tag{3.127}$$

Let $\delta \in (0, 1)$. By (3.99), (3.116), and (3.127),

$$\bar{q}\gamma_0\bar{N} \ge \|x_{q\bar{N}} - x_{(q+1)\bar{N}}\|$$
$$\ge \|z_\delta - x_{q\bar{N}}\| - \|z_\delta - x_{(q+1)\bar{N}}\|$$
$$\ge (\|z_\delta - x_{q\bar{N}}\|^2 - \|z_\delta - x_{(q+1)\bar{N}}\|^2)(8M)^{-1}. \tag{3.128}$$

In view of (3.119) and (3.128),

$$8\bar{q}\gamma_0\bar{N}M \geq \sum_{i=q\bar{N}}^{(q+1)\bar{N}-1} (\|z_\delta - x_i\|^2 - \|z_\delta - x_{i+1}\|^2)$$
$$\geq \sum_{i=q\bar{N}}^{(q+1)\bar{N}-1} \bar{c}\Delta\lambda_{i+1}^2 - 2\delta\bar{q}(8M+1)\bar{N}.$$

Since δ is any element of the interval (0, 1) we conclude that

$$8\bar{q}\gamma_0\bar{N}M \geq \bar{c}\Delta \sum_{i=q\bar{N}}^{(q+1)\bar{N}-1} \lambda_{i+1}^2$$

and

$$\lambda_{i+1} \leq (8\bar{q}\gamma_0\bar{N}M\bar{c}^{-1}\Delta^{-1})^{1/2},\ i = q\bar{N}, \ldots, (q+1)\bar{N}-1. \tag{3.129}$$

Set

$$\Delta_1 = (8\bar{q}\gamma_0\bar{N}M\bar{c}^{-1}\Delta^{-1})^{1/2}. \tag{3.130}$$

Let $i \in \{q\bar{N}, \ldots, (q+1)\bar{N}-1\}$. By (3.105), (3.107), (3.108), (3.129), and (3.130), for every $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and every $j = 0, \ldots, p(t)-1$,

$$\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\| = \|y_j^{(i,t)} - y_{j+1}^{(i,t)}\|$$
$$\leq \lambda_{i+1} \leq (8\bar{q}\gamma_0\bar{N}M\bar{c}^{-1}\Delta^{-1})^{1/2} = \Delta_1 \tag{3.131}$$

and

$$y_j^{(i,t)} \in F_{\Delta_1}(P_{t_{j+1}}). \tag{3.132}$$

It follows from (3.11), (3.104), (3.106), and (3.131) that for every $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and every $j = 0, \ldots, p(t)$,

$$\|x_i - y_j^{(i,t)}\| \leq \Delta_1 p(t) \leq \Delta_1\bar{q}, \tag{3.133}$$

$$\|x_i - y_{i,t}\| \leq \Delta_1\bar{q}, \tag{3.134}$$

$$\|x_i - x_{i+1}\| = \|x_i - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t}\| \leq \Delta_1\bar{q}. \tag{3.135}$$

In view of (3.135), for each $i_1, i_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$\|x_{i_1} - x_{i_2}\| \le \bar{N}\bar{q}\Delta_1. \tag{3.136}$$

Let $k \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$ and $s \in \{1, \dots, m\}$. By (3.26), there exist

$$j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1} \tag{3.137}$$

such that

$$s \in \{t_1, \dots, t_{p(t)}\}. \tag{3.138}$$

In view of (3.138), there exists an integer l such that

$$l \in \{0, \dots, p(t) - 1\},\ s = t_{l+1}. \tag{3.139}$$

It follows from (3.131), (3.137), and (3.139) that

$$\|y_l^{(j,t)} - P_s(y_l^{(j,t)})\| \le \Delta_1. \tag{3.140}$$

By (3.133),

$$\|x_j - y_l^{(j,t)}\| \le \Delta_1\bar{q}. \tag{3.141}$$

It follows from (3.29), (3.140), and (3.141) that

$$\begin{aligned} &\|x_j - P_s(x_j)\| \\ &\le \|x_j - y_l^{(j,t)}\| + \|y_l^{(j,t)} - P_s(y_l^{(j,t)})\| + \|P_s(y_l^{(j,t)}) - P_s(x_j)\| \\ &\le \Delta_1 + 2\|x_j - y_l^{(j,t)}\| \le \Delta_1(2\bar{q} + 1). \end{aligned} \tag{3.142}$$

By (3.136) and (3.137),

$$\|x_k - x_j\| \le \Delta_1\bar{q}\bar{N}.$$

Together with (3.29), (3.130), (3.141), and (3.142) this implies that

$$\begin{aligned} &\|x_k - P_s(x_k)\| \\ &\le \|x_k - x_j\| + \|x_j - P_s(x_j)\| + \|P_s(x_j) - P_s(x_k)\| \\ &\le \Delta_1(2\bar{q} + 1) + 2\|x_k - x_j\| \\ &\le \Delta_1(2\bar{q} + 1) + 2\Delta_1\bar{q}\bar{N} = \Delta_1(2\bar{q}\bar{N} + 2\bar{q} + 1) \le \epsilon. \end{aligned} \tag{3.143}$$

In view of (3.143),

$$x_k \in F_\epsilon(P_s),\ s = 1, \dots, m,$$
$$x_k \in F_\epsilon,\ k = q\bar{N}, \dots, (q+1)\bar{N}$$

and all integers $q \geq q_0$. Therefore

$$x_k \in F_\epsilon$$

for all integers

$$k \geq q_0\bar{N} = \bar{N}(1 + 4M^2\bar{c}^{-1}\Delta^{-1}\epsilon^{-4}(8\bar{q}\bar{N})^6(\bar{c}\Delta)^{-2}).$$

Theorem 3.2 is proved.

3.5 Proof of Theorem 3.3

Theorem 3.3 is deduced from Theorems 2.9 and 3.2. Let $Y = X$, $\rho(y, z) = \|y - z\|$, $y, z \in X$, $\mathfrak{A}$ be the set of all mappings S defined on the set of natural numbers such that

$$S(i) = P_{\Omega_i, w_i},\ i = 1, 2, \dots,$$

where

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies

$$(\Omega_{i+\bar{N}}, w_{i+\bar{N}}) = (\Omega_i, w_i) \text{ for all integers } i \geq 1$$

and

$$\{1, \dots, m\} \subset \cup_{i=1}^{\bar{N}}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}).$$

Set

$$F = F_{\epsilon_0/4}.$$

Theorem 3.2 implies that property (P6) holds.

Let $i \geq 0$ be an integer. By (3.40),

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), \delta). \tag{3.144}$$

By (3.20) and (3.144), there exist vectors

$$(y_{i,t}, \alpha_{i,t}) \in A_0(x_i, t, \delta),\ t \in \Omega_{i+1} \tag{3.145}$$

such that

$$\|x_{i+1} - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\| \leq \delta, \tag{3.146}$$

$$\lambda_{i+1} = \max\{\alpha_{i,t} :\ t \in \Omega_{i+1}\}. \tag{3.147}$$

It follows from (3.19) and (3.145) that for every index vector

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$$

there exists a finite sequence $\{y_j^{(i,t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(i,t)} = x_i,\ y_{p(t)}^{(i,t)} = y_{i,t} \tag{3.148}$$

for every integer $j = 1, \ldots, p(t)$,

$$\|y_j^{(i,t)} - P_{t_j}(y_{j-1}^{(i,t)})\| \leq \delta, \tag{3.149}$$

$$\alpha_{i,t} = \max\{\|y_{j+1}^{(i,t)} - y_j^{(i,t)}\| :\ j = 0, \ldots, p(t) - 1\}. \tag{3.150}$$

Proposition 2.8, (3.17), (3.29), (3.148), and (3.149) imply that for every index vector $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$,

$$\|y_{i,t} - P[t](x_0)\| \leq \|y_{p(t)}^{(i,t)} - P[t](y_0^{(i,t)})\| \leq p(t)\delta \leq \bar{q}\delta.$$

By the relation above, (3.36) and (3.146),

$$\begin{aligned}
&\|x_{i+1} - P_{\Omega_{i+1}, w_{i+1}}(x_i)\| \\
&\leq \|x_{i+1} - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\| + \|\sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t} - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) P[t](x_0)\| \\
&\leq (\bar{q} + 1)\delta \leq 4^{-1}\epsilon_0(Q(2\bar{N} + 1))^{-1}.
\end{aligned} \tag{3.151}$$

Theorem 2.9 and (3.151) imply that for all integer $i \geq Q$,

$$B(x_i, \epsilon_0/4) \cap F_{\epsilon_0/4} \neq \emptyset. \tag{3.152}$$

Let $i \geq Q$ be an integer. In view of (3.152), there exists

$$\xi \in B(x_i, \epsilon_0/4) \cap F_{\epsilon_0/4}.$$

Then for all $s = 1, \dots, m$,

$$\begin{aligned} &\|x_i - P_s(x_i)\| \\ &\leq \|x_i - \xi\| + \|\xi - P_s(\xi)\| + \|P_s(\xi) - P_s(x_i)\| \\ &\leq 2\|x_i - \xi\| + \epsilon_0/4 < \epsilon_0. \end{aligned}$$

Theorem 3.3 is proved.

3.6 The Second Problem

Recall that $M_* > 1$. We suppose that the following assumption holds.

(A2) For each $M > 0$ and each $\gamma > 0$ there exists $\delta > 0$ such that for each $i \in \{1, \dots, m\}$, each

$$z \in B(0, M) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(0, M)$ satisfying $\|x - P_i(x)\| \geq \gamma$, the inequality

$$\|P_i(x) - z\| \leq \|x - z\| - \delta$$

is true.

In this chapter we prove the following three results: Theorem 3.4 which shows that the inexact dynamic string-averaging method generates approximate solutions if perturbations are summable, Theorem 3.5 which establishes that the exact dynamic string-averaging method generates approximate solutions, and Theorem 3.6 which demonstrates that the inexact dynamic string-averaging method generates approximate solutions if the perturbations are small enough.

Theorem 3.4 *Let*

$$M \geq M_*, \ \epsilon \in (0, 1)$$

and let a sequence $\{\epsilon_i\}_{i=1}^{\infty} \subset (0, \infty)$ *satisfy*

$$\Lambda := \sum_{i=1}^{\infty} \epsilon_i < \infty. \tag{3.153}$$

Then there exists a natural number $Q > 0$ *such that for each*

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{3.154}$$

which satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1, \dots, t_{p(t)}\}) \tag{3.155}$$

and each pair of sequences $\{x_i\}_{i=1}^{\infty} \subset X$ *and* $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *which satisfies*

$$x_0 \in B(0, M) \tag{3.156}$$

and

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \epsilon_i), \ i = 1, 2, \dots, \tag{3.157}$$

the following inequality holds:

$$Card(\{i \in \{0, 1, \dots\} : \ x_i \notin \tilde{F}_\epsilon\}) \le Q.$$

Theorem 3.5 *Assume that for each* $x, y \in X$ *and each* $i \in \{1, \dots, m\}$,

$$\|P_i(x) - P_i(y)\| \le \|x - y\|. \tag{3.158}$$

Let $M \ge M_*$, $\epsilon \in (0, 1)$. *Then there exists a constant* $Q > 0$ *such that for each*

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfying for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1, \dots, t_{p(t)}\})$$

and

$$(\Omega_{i+\bar{N}}, w_{i+\bar{N}}) = (\Omega_i, w_i) \text{ for all integers } i \ge 0$$

and each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ *which satisfies*

$$x_0 \in B(0, M),$$

$$x_i = P_{\Omega_{i+1}, w_{i+1}}(x_i)$$

for each integer $i \ge 0$, *the inclusion*

$$x_i \in F_\epsilon$$

holds for all integers $i \ge Q$.

Theorem 3.6 *Assume that for each $x, y \in X$ and each $i \in \{1, \dots, m\}$,*

$$\|P_i(x) - P_i(y)\| \le \|x - y\|.$$

Let $M \ge M_*$, $r_0 \in (0, 1)$,

$$F_{r_0} \subset B(0, M),$$

$\epsilon_0 \in (0, 1)$. *Then there exists $Q > 0$, $\delta > 0$ such that for each*

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfying for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\})$$

and

$$(\Omega_{i+\bar{N}}, w_{i+\bar{N}}) = (\Omega_i, w_i) \text{ for all integers } i \ge 0$$

and each pair of sequences $\{x_i\}_{i=1}^{\infty} \subset X$ and $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ which satisfies

$$x_0 \in B(0, M),$$
$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta)$$

for all natural numbers i, the inclusion

$$x_i \in F_{\epsilon_0}$$

holds for all integers $i \ge Q$.

3.7 Proof of Theorem 3.4

By (A1), for every $\delta > 0$ there exists

$$z_\delta \in B(0, M_*) \tag{3.159}$$

such that

$$B(z_\delta, \delta) \cap \mathrm{Fix}(P_i) \ne \emptyset, \;\; i = 1, \dots, m. \tag{3.160}$$

In view of (3.160), for each $\delta > 0$ and each $i \in \{1, \dots, m\}$ there exists

$$z_{\delta,i} \in B(z_\delta, \delta) \cap \mathrm{Fixp}(P_i). \tag{3.161}$$

Set

$$\gamma_0 = \epsilon(\bar{N} + 1)^{-1}(\bar{q} + 1)^{-1}. \tag{3.162}$$

By (A2), there exists $\gamma \in (0, \gamma_0)$ such that the following property holds:

(P2) for each $i \in \{1, \dots, m\}$, each

$$z \in B(0, 3M + 1 + \Lambda) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(0, 3M + 1 + \Lambda(\bar{q} + 1))$ satisfying $\|x - P_i(x)\| \ge \gamma_0/2$, the inequality

$$\|P_i(x) - z\| \le \|x - z\| - \gamma$$

is true.

In view of (3.153), there exists a natural number n_0 such that

$$\epsilon_i < \gamma/4 \text{ for all integers } i \ge n_0. \tag{3.163}$$

Set

$$Q = n_0 + 2\bar{N}(\Delta\gamma)^{-1}(M + (\bar{q} + 1)\Lambda). \tag{3.164}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{3.165}$$

satisfies (3.155) for each natural number j, $\{x_i\}_{i=0}^{\infty} \subset X$ and $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfies

$$x_0 \in B(0, M) \tag{3.166}$$

and

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \epsilon_i), \ \ i = 1, 2, \dots, \tag{3.167}$$

Let $i \ge 0$ be an integer. By (3.167),

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), \epsilon_{i+1}). \tag{3.168}$$

By (3.20) and (3.168), there exist vectors

$$(y_{i,t}, \alpha_{i,t}) \in A_0(x_i, t, \epsilon_{i+1}),\ t \in \Omega_{i+1} \tag{3.169}$$

such that

$$\|x_{i+1} - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\| \le \epsilon_{i+1}, \tag{3.170}$$

$$\lambda_{i+1} = \max\{\alpha_{i,t} :\ t \in \Omega_{i+1}\}. \tag{3.171}$$

It follows from (3.19) and (3.169) that for every index vector

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$$

there exists a finite sequence $\{y_j^{(i,t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(i,t)} = x_i,\ y_{p(t)}^{(i,t)} = y_{i,t}, \tag{3.172}$$

for every integer $j = 1, \dots, p(t)$,

$$\|y_j^{(i,t)} - P_{t_j}(y_{j-1}^{(i,t)})\| \le \epsilon_{i+1}, \tag{3.173}$$

$$\alpha_{i,t} = \max\{\|y_{j+1}^{(i,t)} - y_j^{(i,t)}\| :\ j = 0, \dots, p(t) - 1\}. \tag{3.174}$$

Set

$$\epsilon_0 = 0. \tag{3.175}$$

Let $\delta > 0$. By (3.159) and (3.166),

$$\|z_\delta - x_0\| \le 2M. \tag{3.176}$$

Let $i \ge 0$ be an integer,

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1},\ j \in \{0, \dots, p(t) - 1\}. \tag{3.177}$$

Relations (3.7), (3.161), and (3.173) imply that

$$\begin{aligned}
\|z_\delta - y_{j+1}^{(i,t)}\| &\le \|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\| + \|P_{t_{j+1}}(y_j^{(i,t)}) - y_{j+1}^{(i,t)}\| \\
&\le \|z_\delta - z_{\delta,t_{j+1}}\| + \|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\| + \epsilon_{i+1} \\
&\le \delta + \epsilon_{i+1} + \|z_{\delta,t_{j+1}} - y_j^{(i,t)}\| \\
&\le \|z_\delta - y_j^{(i,t)}\| + 2\delta + \epsilon_{i+1}
\end{aligned}$$

and

$$\|z_\delta - y_{j+1}^{(i,t)}\| \le \|z_\delta - y_j^{(i,t)}\| + 2\delta + \epsilon_{i+1}. \tag{3.178}$$

It follows from (3.17), (3.172), and (3.178) that for all $j = 0, \ldots, p(t)$,

$$\begin{aligned}\|z_\delta - y_j^{(i,t)}\| &\le \|z_\delta - y_0^{(i,t)}\| + j(2\delta + \epsilon_{i+1}) \\ &= \|z_\delta - x_i\| + j(2\delta + \epsilon_{i+1}) \\ &\le \|z_\delta - x_i\| + p(t)(2\delta + \epsilon_{i+1}) \\ &\le \|z_\delta - x_i\| + \bar{q}(2\delta + \epsilon_{i+1}). \end{aligned} \tag{3.179}$$

By (3.172) and (3.179),

$$\|z_\delta - y_{i,t}\| = \|z_\delta - y_{p(t)}^{(i,t)}\| \le \|z_\delta - x_i\| + q(2\delta + \epsilon_{i+1}). \tag{3.180}$$

By (3.11), (3.170), (3.180), and the convexity of the norm,

$$\begin{aligned}\|z_\delta - x_{i+1}\| &\le \|z_\delta - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\| + \|\sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t} - x_{i+1}\| \\ &\le \sum_{t \in \Omega_{i+1}} w_{i+1}(t)\|z_\delta - y_{i,t}\| + \epsilon_{i+1} \\ &\le \|z_\delta - x_i\| + (\bar{q} + 1)(2\delta + \epsilon_{i+1}). \end{aligned} \tag{3.181}$$

By induction we show that for all integers $i \ge 0$,

$$\|z_\delta - x_i\| \le 2M + 2(\bar{q} + 1)\delta i + (\sum_{j=0}^{i} \epsilon_j)(\bar{q} + 1). \tag{3.182}$$

In view of (3.175) and (3.176), inequality (3.182) is true for $i = 0$.

Assume that $i \ge 0$ is an integer and that (3.182) holds. By (3.181) and (3.182),

$$\begin{aligned}\|z_\delta - x_{i+1}\| &\le \|z_\delta - x_i\| + (\bar{q} + 1)(2\delta + \epsilon_{i+1}) \\ &\le 2M + 2(\bar{q} + 1)\delta(i + 1) + (\sum_{j=0}^{i+1} \epsilon_j)(\bar{q} + 1). \end{aligned}$$

Therefore by induction we showed that (3.182) holds for all integers $i \ge 0$. It follows from (3.179) and (3.182) that for all integers $i \ge 0$, all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \ldots, p(t)$,

$$\|z_\delta - y_j^{(i,t)}\| \le \|z_\delta - x_i\| + \bar{q}(2\delta + \epsilon_{i+1})$$

$$\le 2M + 2(\bar{q}+1)\delta(i+1) + (\sum_{j=0}^{i+1} \epsilon_j)(\bar{q}+1). \quad (3.183)$$

Let n be a natural number. By (3.153), (3.159), (3.182), and (3.183), for all integers $i = 0, \dots, n$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t)$,

$$\|x_i\| \le \|z_\delta\| + \|x_i - z_\delta\| \le 3M + 2(\bar{q}+1)\delta n + \Lambda(\bar{q}+1),$$

$$\|y_j^{(i,t)}\| \le \|z_\delta\| + \|y_j^{(i,t)} - z_\delta\| \le 3M + 2(\bar{q}+1)\delta(n+1) + \Lambda(\bar{q}+1).$$

Since the relation above holds for any $\delta > 0$ we conclude that for all integers $i \ge 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$, and all $j = 0, \dots, p(t)$,

$$\|x_i\| \le 3M + \Lambda(\bar{q}+1), \quad (3.184)$$

$$\|y_j^{(i,t)}\| \le 3M + +\Lambda(\bar{q}+1). \quad (3.185)$$

Set

$$E_0 = \{i \in \{n_0, n_0+1, \dots\} : \ \lambda_{i+1} \ge \gamma_0\}, \quad (3.186)$$

$$E_1 = \{n_0, n_0+1, \dots, \} \setminus E_0.$$

Let

$$i \in E_0.$$

By (3.171) and (3.186),

$$\lambda_{i+1} \ge \gamma_0$$

and there exists

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \Omega_{i+1}$$

such that

$$\alpha_{i,\tau} = \lambda_{i+1} \ge \gamma_0. \quad (3.187)$$

By (3.174) and (3.187), there exists

$$j_0 \in \{1, \dots, p(\tau) - 1\} \quad (3.188)$$

such that

$$\|y_{j_0+1}^{(i,\tau)} - y_{j_0}^{(i,\tau)}\| = \alpha_{i,\tau} \geq \gamma_0. \tag{3.189}$$

It follows from (3.173) and (3.189) that

$$\begin{aligned} &\|y_{j_0}^{(i,\tau)} - P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)})\| \\ &\geq \|y_{j_0+1}^{(i,\tau)} - y_{j_0}^{(i,\tau)}\| - \|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - y_{j_0+1}^{(i,\tau)}\| \geq \gamma_0 - \epsilon_{i+1}. \end{aligned} \tag{3.190}$$

In view of (3.163), (3.186), and (3.190),

$$\|y_{j_0}^{(i,\tau)} - P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)})\| \geq \gamma_0 - \epsilon_{i+1} \geq \gamma_0 - \gamma/4 \geq \gamma_0/2. \tag{3.191}$$

Let $\delta \in (0, 1)$. Property (P2), (3.160), (3.161), (3.185), and (3.191) imply that

$$\|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - z_{\delta,\tau_{j_0+1}}\| \leq \|y_{j_0}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| - \gamma. \tag{3.192}$$

By (3.161), (3.163), (3.173), (3.186), and (3.192),

$$\begin{aligned} \|y_{j_0+1}^{(i,\tau)} - z_\delta\| &\leq \|y_{j_0+1}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| + \|z_{\delta,\tau_{j_0+1}} - z_\delta\| \\ &\leq \|y_{j_0+1}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| + \delta \\ &\leq \|y_{j_0+1}^{(i,\tau)} - P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)})\| + \|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - z_{\delta,\tau_{j_0+1}}\| + \delta \\ &\leq \epsilon_{i+1} + \|y_{j_0}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| - \gamma + \delta \\ &\leq \gamma/4 - \gamma + \delta + \|y_{j_0}^{(i,\tau)} - z_\delta\| + \delta \\ &\leq 2\delta - 3\gamma/4 + \|y_{j_0}^{(i,\tau)} - z_\delta\|, \end{aligned}$$

$$\|y_{j_0+1}^{(i,\tau)} - z_\delta\| \leq \|y_{j_0}^{(i,\tau)} - z_\delta\| - 3\gamma/4 + 2\delta. \tag{3.193}$$

By (3.17), (3.172), (3.178), (3.188), and (3.193),

$$\begin{aligned} &\|z_\delta - x_i\| - \|z_\delta - y_{i,\tau}\| \\ &= \sum_{j=0}^{p(\tau)-1} [\|z_\delta - y_j^{(i,\tau)}\| - \|z_\delta - y_{j+1}^{(i,\tau)}\|] \\ &\geq \|y_{j_0}^{(i,\tau)} - z_\delta\| - \|y_{j_0+1}^{(i,\tau)} - z_\delta\| - (p(\tau) - 1)(2\delta + \epsilon_{i+1}) \\ &\geq 3\gamma/4 - 2\delta - (p(\tau) - 1)(2\delta + \epsilon_{i+1}) \\ &\geq 3\gamma/4 - 2\delta - (\bar{q} - 1)(2\delta + \epsilon_{i+1}). \end{aligned} \tag{3.194}$$

It follows from (3.11), (3.17), (3.18), (3.170), (3.180), (3.194), and the convexity of the function $\|\cdot\|$ that

$$\|z_\delta - x_{i+1}\|$$
$$\leq \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t}\| + \|\sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t} - x_{i+1}\|$$
$$\leq \sum_{t\in\Omega_{i+1}} w_{i+1}(t)\|z_\delta - y_{i,t}\| + \epsilon_{i+1}$$
$$\leq \epsilon_{i+1} + \|z_\delta - x_i\| + \sum_{t\in\Omega_{i+1}} w_{i+1}(t)[\|z_\delta - y_{i,t}\| - \|z_\delta - x_i\|]$$
$$\leq \epsilon_{i+1} + \|z_\delta - x_i\| + w_{i+1}(\tau)[\|z_\delta - y_{i,\tau}\| - \|z_\delta - x_i\|]$$
$$+ \sum\{w_{i+1}(t)[\|z_\delta - y_{i,t}\| - \|z_\delta - x_i\|]:\ t \in \Omega_{i+1}\setminus\{\tau\}\}$$
$$\leq \epsilon_{i+1} + \|z_\delta - x_i\| + w_{i+1}(\tau)(-3\gamma/4 + 2\delta + (\bar{q}-1)(2\delta + \epsilon_{i+1})) + \bar{q}(2\delta + \epsilon_{i+1})$$
$$\leq \|z_\delta - x_i\| - 3\Delta\gamma/4 + 2\bar{q}(2\delta + \epsilon_{i+1}),$$
$$\|z_\delta - x_{i+1}\| \leq \|z_\delta - x_i\| - 3\Delta\gamma/4 + 2\bar{q}(2\delta + \epsilon_{i+1}). \tag{3.195}$$

By (3.152), (3.159), (3.175), (3.181), (3.184), (3.186), and (3.195), for every integer $n > n_0$,

$$4M + \Lambda(\bar{q}+1) \geq \|z_\delta - x_{n_0}\|$$
$$\geq \|z_\delta - x_{n_0}\| - \|z_\delta - x_n\|$$
$$\geq \sum_{i=n_0}^{n-1}(\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\|)$$
$$= \sum_{i\in E_0\cap[0,n-1]}(\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\|)$$
$$+ \sum_{i\in E_1\cap[0,n-1]}(\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\|)$$
$$\geq \mathrm{Card}(E_0\cap[0,n-1])(3\Delta\gamma/4) - 4\bar{q}\delta n - 2\bar{q}\sum_{i=0}^{n}\epsilon_i - (\bar{q}+1)\sum_{i\in E_1\cap[0,n-1]}(2\delta + \epsilon_{i+1})$$
$$\geq \mathrm{Card}(E_0\cap[0,n-1])(3\Delta\gamma/4) - \Lambda(3\bar{q}+1) - 4\bar{q}\delta n - 2(\bar{q}+1)\delta n.$$

Since δ is any element of the interval $(0, 1)$ we conclude that

$$(3\Delta\gamma/4)\mathrm{Card}(E_0 \cap [0, n-1]) \le 4M + 4\Lambda(\bar{q}+1),$$

$$\mathrm{Card}(E_0 \cap [0, n-1]) \le 2\Delta^{-1}\gamma^{-1}(4M + 4\Lambda(\bar{q}+1)).$$

Since n is any natural number satisfying $n > n_0$ we conclude that

$$\mathrm{Card}(E_0) \le 2\Delta^{-1}\gamma^{-1}(4M + 4\Lambda(\bar{q}+1)). \tag{3.196}$$

Assume that a natural number i satisfies

$$i \ge n_0,\ \lambda_{i+1} < \gamma_0. \tag{3.197}$$

Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$. By (3.171), (3.173), (3.174), and (3.197), for all $j = 0, \dots, p(t)-1$,

$$\begin{aligned}\gamma_0 &> \|y_{j+1}^{(i,t)} - y_j^{(i,t)}\| \\ &\ge \|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\| - \|y_{j+1}^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\| \\ &\ge \|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\| - \epsilon_{i+1}.\end{aligned} \tag{3.198}$$

In view of (3.163), (3.197), and (3.198), for all $j = 0, \dots, p(t)-1$,

$$\|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\| < 2\gamma_0 \tag{3.199}$$

and

$$y_j^{(i,t)} \in F_{2\gamma_0}(P_{t_{j+1}}). \tag{3.200}$$

Relations (3.17), (3.172), (3.198), and (3.200) imply that for all $j = 0, \dots, p(t)$,

$$\|x_i - y_j^{(i,t)}\| \le j\gamma_0 \le \bar{q}\gamma_0 \tag{3.201}$$

and if $j < p(t)$, then

$$x_i \in \tilde{F}_{(\bar{q}+1)\gamma_0}(P_{t_{j+1}}).$$

Therefore

$$x_i \in \tilde{F}_{(\bar{q}+1)\gamma_0}(P_{t_s}),\ s = 1, \dots, p(t),\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}. \tag{3.202}$$

By (3.172) and (3.201),

$$\|x_i - y_{i,t}\| \le \bar{q}\gamma_0 \text{ for all } t \in \Omega_{i+1}. \tag{3.203}$$

It follows from (3.11), (3.163), (3.170), (3.197), (3.203), and the convexity of the norm that

$$\begin{aligned}
&\|x_{i+1} - x_i\| \\
&\le \|x_{i+1} - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t}\| + \|\sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t} - x_i\| \\
&\le \epsilon_{i+1} + \|\sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t} - x_i\| \\
&\le \epsilon_{i+1} + \sum_{t\in\Omega_{i+1}} w_{i+1}(t)\|y_{i,t} - x_i\| \\
&\le \epsilon_{i+1} + \gamma_0\bar{q} < \gamma_0(\bar{q}+1), \\
&\|x_{i+1} - x_i\| < \gamma_0(\bar{q}+1). \qquad (3.204)
\end{aligned}$$

Thus we have shown that the following property holds:

(P3) if a natural number $i \ge n_0$ satisfies $\lambda_{i+1} < \gamma_0$, then (3.202) and (3.204) hold.

Set

$$E_2 = \{i \in \{n_0, n_0+1, \dots\} : [i, i+\bar{N}-1] \cap E_0 \ne \emptyset\}. \qquad (3.205)$$

By (3.196) and (3.205),

$$\begin{aligned}
&\mathrm{Card}(E_2) \le \bar{N}\mathrm{Card}(E_0) \\
&\le 8\bar{N}\Delta^{-1}\gamma^{-1}(M + (\bar{q}+1)\Lambda). \qquad (3.206)
\end{aligned}$$

Let an integer $j \ge n_0$ satisfy

$$j \notin E_2. \qquad (3.207)$$

Property (P3), (3.186), (3.205), and (3.207) imply that

$$[j, j+\bar{N}-1] \cap E_0 = \emptyset,$$

for all $i = j, \dots, j+\bar{N}-1$,

$$\lambda_{i+1} < \gamma_0$$

and that (3.202) and (3.204) hold. It follows from (3.204) which holds for each $i \in \{j, \dots, j+\bar{N}-1\}$ that for each pair of integers $i_1, i_2 \in \{j, \dots, j+\bar{N}\}$,

$$\|x_{i_1} - x_{i_2}\| \le (\bar{q}+1)\bar{N}\gamma_0. \qquad (3.208)$$

By (3.155), (3.202) which holds for each $i \in \{j, \dots, j + \bar{N} - 1\}$ and (3.208),

$$x_j \in \tilde{F}_{(\bar{q}+1)\gamma_0(\bar{N}+1)}(P_s),$$
$$s \in \cup_{i=j}^{j+\bar{N}-1} \cup \{\{t_1, \dots, t_{p(t)}\} : t = \{t_1, \dots, t_{p(t)}\} \in \Omega_{i+1}\} = \{1, \dots, m\}.$$

In view of the relation above and (3.162),

$$x_j \in \tilde{F}_{(\bar{q}+1)\gamma_0(\bar{N}+1)} = \tilde{F}_\epsilon$$

for all integers $j \geq n_0$ such that $j \notin E_2$. Together with (3.165) and (3.206) this implies that

$$\begin{aligned}
&\mathrm{Card}(\{j \in \{0, 1, \dots\} : \ x_j \notin \tilde{F}_\epsilon\}) \\
&\leq n_0 + \mathrm{Card}(E_2) \\
&\leq n_0 + 8\bar{N}\Delta^{-1}(M + \Lambda(\bar{q} + 1))\gamma^{-1} = Q.
\end{aligned}$$

Theorem 3.4 is proved.

3.8 Proof of Theorem 3.5

Set

$$\epsilon_0 = \epsilon(2\bar{q} + 1)^{-1}(\bar{N} + 1)^{-1}. \tag{3.209}$$

By (3.8), (3.11), (3.12), and (3.158), for each integer $(\Omega, w) \in \mathcal{M}_*$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega$ and all $x, y \in X$,

$$\|P[t](x) - P[t](y)\| \leq \|x - y\|, \tag{3.210}$$

$$\begin{aligned}
&\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| \\
&= \|\sum_{t\in\Omega} w(t)P[t](x) - \sum_{t\in\Omega} w(t)P[t](y)\| \leq \|x - y\|.
\end{aligned} \tag{3.211}$$

By (A1), for every $\delta > 0$ there exists

$$z_\delta \in B(0, M_*) \tag{3.212}$$

such that

$$B(z_\delta, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset, \ i = 1, \dots, m. \tag{3.213}$$

In view of (3.213), for each $\delta > 0$ and each $i \in \{1, \dots, m\}$ there exists

$$z_{\delta,i} \in B(z_\delta, \delta) \cap \mathrm{Fixp}(P_i). \tag{3.214}$$

By (A2), there exists $\epsilon_1 \in (0, \epsilon_0)$ such that the following property holds:

(P4) for each $i \in \{1, \dots, m\}$, each

$$z \in B(0, 3M + 1) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(0, 3M + 1)$ satisfying $\|x - P_i(x)\| \ge \epsilon_0$, the inequality

$$\|P_i(x) - z\| \le \|x - z\| - \epsilon_1$$

is true.

Set

$$\gamma_0 = \epsilon_1 (2\bar{N} + 1)^{-1} (8\bar{q})^{-1} \Delta. \tag{3.215}$$

By (A2), there exists $\gamma \in (0, \gamma_0)$ such that the following property holds:

(P5) for each $i \in \{1, \dots, m\}$, each

$$z \in B(0, 3M + 1) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(0, 3M + 1)$ satisfying $\|x - P_i(x)\| \ge \gamma_0/2$, the inequality

$$\|P_i(x) - z\| \le \|x - z\| - \gamma$$

is true.

Set

$$Q = \bar{N}((\Delta\gamma)^{-1} 2M + 1). \tag{3.216}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{3.217}$$

satisfies for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}), \tag{3.218}$$

$$(\Omega_{i+\bar{N}}, w_{i+\bar{N}}) = (\Omega_i, w_i) \text{ for all integers } i \ge 0 \tag{3.219}$$

and $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies

$$x_0 \in B(0, M),$$
$$x_{i+1} = P_{\Omega_{i+1}, w_{i+1}}(x_i) \text{ for all integers } i \geq 0. \tag{3.220}$$

Set

$$T = \prod_{i=1}^{\bar{N}} P_{\Omega_i, w_i} = P_{\Omega_{\bar{N}}, w_{\bar{N}}} \cdots P_{\Omega_1, w_1}. \tag{3.221}$$

Let $i \geq 0$ be an integer. By (3.8), (3.12), (3.19), (3.20), and (3.220), there exists $\lambda_{i+1} \geq 0$ such that

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), 0). \tag{3.222}$$

By (3.20) and (3.222), there exist vectors

$$(y_{i,t}, \alpha_{i,t}) \in A_0(x_i, t, 0), \ t \in \Omega_{i+1} \tag{3.223}$$

such that

$$x_{i+1} = \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}, \tag{3.224}$$

$$\lambda_{i+1} = \max\{\alpha_{i,t} : \ t \in \Omega_{i+1}\}. \tag{3.225}$$

It follows from (3.19) and (3.223) that for every index vector

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$$

there exists a finite sequence $\{y_j^{(i,t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(i,t)} = x_i, \ y_{p(t)}^{(i,t)} = y_{i,t}, \tag{3.226}$$

for every integer $j = 1, \ldots, p(t)$,

$$y_j^{(i,t)} = P_{t_j}(y_{j-1}^{(i,t)}), \tag{3.227}$$

$$\alpha_{i,t} = \max\{\|y_{j+1}^{(i,t)} - y_j^{(i,t)}\| : \ j = 0, \ldots, p(t) - 1\}. \tag{3.228}$$

Let $\delta > 0$. By (3.212) and (3.220),

$$\|z_\delta - x_0\| \leq 2M. \tag{3.229}$$

Let $i \geq 0$ be an integer, $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$, $j = 0, \ldots, p(t) - 1$. In view of (3.7), (3.214), and (3.227),

$$\|z_\delta - y_{j+1}^{(i,t)}\| \leq \|z_\delta - z_{\delta,t_{j+1}}\| + \|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\|$$
$$\leq \delta + \|z_{\delta,t_{j+1}} - y_j^{(i,t)}\|$$
$$\leq 2\delta + \|z_\delta - y_j^{(i,t)}\|. \quad (3.230)$$

By (3.17), (3.226), and (3.230), for all $j = 0, \ldots, p(t)$,

$$\|z_\delta - y_j^{(i,t)}\| \leq \|z_\delta - y_0^{(i,t)}\| + 2\delta j$$
$$\leq \|z_\delta - x_i\| + 2\delta j \leq \|z_\delta - x_i\| + 2\delta \bar{q}. \quad (3.231)$$

Relations (3.226) and (3.231) imply that

$$\|z_\delta - y_{i,t}\| = \|z_\delta - y_{p(t)}^{(i,t)}\| \leq \|z_\delta - x_i\| + 2\delta \bar{q}. \quad (3.232)$$

By (3.11), (3.224), (3.232), and the convexity of the norm,

$$\|z_\delta - x_{i+1}\| = \|z_\delta - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\|$$
$$\leq \sum_{t \in \Omega_{i+1}} w_{i+1}(t) \|z_\delta - y_{i,t}\|$$
$$\leq \|z_\delta - x_i\| + 2\bar{q}\delta. \quad (3.233)$$

In view of (3.229) and (3.232), for all integers $i \geq 0$,

$$\|z_\delta - x_i\| \leq \|z_\delta - x_0\| + 2\delta \bar{q} i \leq 2M + 2\delta \bar{q} i. \quad (3.234)$$

It follows from (3.231) and (3.234) that for all integers $i \geq 0$, all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \ldots, p(t)$,

$$\|z_\delta - y_j^{(i,t)}\| \leq \|z_\delta - x_i\| + 2\delta \bar{q} \leq 2M + 2\delta \bar{q}(i + 1). \quad (3.235)$$

By (3.212) and (3.234), for all integers $i \geq 0$, all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \ldots, p(t)$,

$$\|x_i\| \leq \|z_\delta\| + \|x_i - z_\delta\| \leq 3M + 2\bar{q}\delta i,$$
$$\|y_j^{(i,t)}\| \leq \|z_\delta\| + \|y_j^{(i,t)} - z_\delta\| \leq 3M + 2\bar{q}\delta(i + 1).$$

Since the relation above holds for any $\delta > 0$ we conclude that for all integers $i \geq 0$, all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$, and all $j = 0, \ldots, p(t)$,

$$\|x_i\| \leq 3M, \ \|y_j^{(i,t)}\| \leq 3M. \tag{3.236}$$

Set

$$E_0 = \{i \in \{0, 1, \ldots\} : \ \lambda_{i+1} \geq \gamma_0\}, \ E_1 = \{0, 1, \ldots, \} \setminus E_0. \tag{3.237}$$

Let $\delta \in (0, 1)$ and

$$i \in E_0. \tag{3.238}$$

By (3.225), (3.237), and (3.238), there exists

$$\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \Omega_{i+1}$$

such that

$$\alpha_{i,\tau} = \lambda_{i+1} \geq \gamma_0. \tag{3.239}$$

By (3.228) and (3.239), there exists

$$j_0 \in \{1, \ldots, p(\tau) - 1\} \tag{3.240}$$

such that

$$\|y_{j_0+1}^{(i,\tau)} - y_{j_0}^{(i,\tau)}\| = \alpha_{i,\tau} \geq \gamma_0. \tag{3.241}$$

It follows from (3.212), (3.214), (3.227), (3.236), (3.241), and property (P5) that

$$\|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - y_{j_0}^{(i,\tau)}\| \geq \gamma_0,$$
$$\|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - z_{\delta,\tau_{j_0+1}}\| \geq \|y_{j_0}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| - \gamma. \tag{3.242}$$

In view of (3.17), (3.214), (3.227), and (3.242),

$$\begin{aligned}\|y_{j_0+1}^{(i,\tau)} - z_\delta\| &\leq \|y_{j_0+1}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| + \delta \\ &= \|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - z_{\delta,\tau_{j_0+1}}\| + \delta \\ &\leq \|y_{j_0}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| - \gamma + \delta \\ &\leq \|y_{j_0}^{(i,\tau)} - z_\delta\| - \gamma + 2\delta.\end{aligned} \tag{3.243}$$

By (3.226), (3.230), (3.240), and (3.243),

$$\begin{aligned}
&\|z_\delta - x_i\| - \|z_\delta - y_{i,\tau}\| \\
&= \|z_\delta - y_0^{(i,\tau)}\| - \|z_\delta - y_{p(\tau)}^{(i,\tau)}\| \\
&= \sum_{j=0}^{p(\tau)-1} [\|z_\delta - y_j^{(i,\tau)}\| - \|z_\delta - y_{j+1}^{(i,\tau)}\|] \\
&\geq \|z_\delta - y_{j_0}^{(i,\tau)}\| - \|z_\delta - y_{j_0+1}^{(i,\tau)}\| - 2\delta(p(\tau) - 1) \\
&\geq \gamma - 2\delta p(\tau) \geq \gamma - 2\delta\bar{q}. \qquad (3.244)
\end{aligned}$$

It follows from (3.11), (3.224), (3.232), (3.244), and the convexity of the function $\|\cdot\|$ that

$$\begin{aligned}
&\|z_\delta - x_{i+1}\| \\
&= \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t) y_{i,t}\| \\
&\leq \sum_{t\in\Omega_{i+1}} w_{i+1}(t)\|z_\delta - y_{i,t}\| \\
&= w_{i+1}(\tau)\|z_\delta - y_{i,\tau}\| \\
&+ \sum\{w_{i+1}(t)\|z_\delta - y_{i,t}\| :\ t \in \Omega_{i+1} \setminus \{\tau\}\} \\
&\leq w_{i+1}(\tau)(\|z_\delta - x_i\| - \gamma + 2\delta\bar{q}) \\
&+ \sum\{w_{i+1}(t)(\|z_\delta - x_i\| + 2\delta\bar{q}) :\ t \in \Omega_{i+1} \setminus \{\tau\}\} \\
&\leq \|z_\delta - x_i\| + 2\delta\bar{q} - \gamma w_{i+1}(\tau) \\
&\leq \|z_\delta - x_i\| + 2\delta\bar{q} - \Delta\gamma.
\end{aligned}$$

Thus

$$\|z_\delta - x_{i+1}\| \leq \|z_\delta - x_i\| + 2\delta\bar{q} - \Delta\gamma \text{ for all } i \in E_0. \qquad (3.245)$$

Let n be a natural number. By (3.229),

$$\begin{aligned}
&2M \geq \|z_\delta - x_0\| \\
&\geq \|z_\delta - x_0\| - \|z_\delta - x_{\bar{N}n}\| \\
&= \sum_{k=0}^{n-1}(\|z_\delta - x_{k\bar{N}}\| - \|z_\delta - x_{(k+1)\bar{N}}\|)
\end{aligned}$$

$$= \sum_{k=0}^{n-1} (\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (\|z_\delta - x_j\| - \|z_\delta - x_{j+1}\|)). \tag{3.246}$$

Set

$$E_2 = \{k \in \{0, 1, \dots\} : \max\{\lambda_{i+1} : i = k\bar{N}, \dots, (k+1)\bar{N}-1\} \geq \gamma_0\}. \tag{3.247}$$

Assume that an integer $k \in [0, n-1]$ satisfies

$$k \in E_2.$$

Then by (3.233), (3.237), (3.245), and (3.247),

$$\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (\|z_\delta - x_j\| - \|z_\delta - x_{j+1}\|)$$
$$= \sum \{\|z_\delta - x_j\| - \|z_\delta - x_{j+1}\| : j \in \{k\bar{N}, \dots, (k+1)\bar{N} - 1\}, \ \lambda_{j+1} \geq \gamma_0\}$$
$$+ \sum \{\|z_\delta - x_j\| - \|z_\delta - x_{j+1}\| : j \in \{k\bar{N}, \dots, (k+1)\bar{N} - 1\}, \ \lambda_{j+1} < \gamma_0\}$$
$$\geq \Delta\gamma - 2\delta\bar{q} - 2\delta\bar{q}\bar{N}. \tag{3.248}$$

It follows from (3.233) and (3.246) that

$$2M \geq \sum \{ \sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (\|z_\delta - x_j\| - \|z_\delta - x_{j+1}\|) : k \in E_2 \cap [0, n-1]\}$$
$$+ \sum \{ \sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (\|z_\delta - x_j\| - \|z_\delta - x_{j+1}\|) : k \in \{0, \dots, n-1\} \setminus E_2\}$$
$$\geq \mathrm{Card}(E_2 \cap [0, n-1])(\Delta\gamma - 2\delta\bar{q}(\bar{N} + 1))$$
$$+ \mathrm{Card}(\{0, \dots, n-1\} \setminus E_2)(-2\delta\bar{q})$$
$$\geq \Delta\gamma \mathrm{Card}(E_2 \cap [0, n-1]) - 2\delta\bar{q}(\bar{N} + 1)n.$$

Since δ is any element of the interval (0, 1) we conclude that

$$\mathrm{Card}(E_2 \cap [0, n-1]) \leq 2M(\Delta\gamma)^{-1}.$$

Since the relation above holds for every natural number n we conclude that

$$\mathrm{Card}(E_2) \leq 2M(\Delta\gamma)^{-1}. \tag{3.249}$$

In view of (3.247) and (3.249), there exists an integer $q_0 \geq 0$ such that

$$q_0 \leq 2M(\Delta\gamma)^{-1} + 1, \ q_0 \notin E_2 \tag{3.250}$$

and

$$\lambda_{i+1} < \gamma_0, \ i = q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1. \tag{3.251}$$

By (3.225), (3.226), and (3.251), for all integers $i = q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t) - 1$,

$$\|y_j^{(i,t)} - y_{j+1}^{(i,t)}\| \leq \gamma_0, \tag{3.252}$$

$$\begin{aligned}\|x_i - y_{i,t}\| &= \|y_0^{(i,t)} - y_{p(t)}^{(i,t)}\| \\ &\leq \sum_{j=0}^{p(t)-1} \|y_j^{(i,t)} - y_{j+1}^{(i,t)}\| \leq \bar{q}\gamma_0.\end{aligned} \tag{3.253}$$

It follows from (3.11), (3.224), and the convexity of the function $\|\cdot\|$ that for all $i = q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1$,

$$\begin{aligned}&\|x_i - x_{i+1}\| \\ &\leq \|x_i - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\| \\ &\leq \sum_{t \in \Omega_{i+1}} w_{i+1}(t)\|x_i - y_{i,t}\| \leq \bar{q}\gamma_0.\end{aligned} \tag{3.254}$$

In view of (3.254),

$$\|x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}}\| \leq \bar{q}\gamma_0\bar{N}. \tag{3.255}$$

By (3.211), (3.219)–(3.221), and (3.255), for each integer $q > q_0$,

$$\begin{aligned}\|x_{q\bar{N}} - x_{(q+1)\bar{N}}\| &= \|T^{q-q_0}(x_{q_0\bar{N}}) - T^{q-q_0}(x_{(q_0+1)\bar{N}})\| \\ &\leq \|x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}}\| \leq \bar{q}\gamma_0\bar{N}.\end{aligned} \tag{3.256}$$

Let $q \geq q_0$ be an integer. In view of (3.255) and (3.256),

$$\|x_{q\bar{N}} - x_{(q+1)\bar{N}}\| \leq \bar{q}\gamma_0\bar{N}. \tag{3.257}$$

Let $\delta \in (0, 1)$. By (3.257),

$$\bar{q}\gamma_0\bar{N} \geq \|x_{q\bar{N}} - x_{(q+1)\bar{N}}\|$$
$$\geq \|z_\delta - x_{q\bar{N}}\| - \|z_\delta - x_{(q+1)\bar{N}}\|. \tag{3.258}$$

In view of (3.233),

$$\bar{q}\gamma_0\bar{N} \geq \sum_{i=q\bar{N}}^{(q+1)\bar{N}-1} (\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\|)$$
$$= \sum\{\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\| : \ i \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}, \ \lambda_{i+1} \geq \epsilon_0\}$$
$$+ \sum\{\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\| : \ i \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}, \ \lambda_{i+1} < \epsilon_0\}$$
$$\geq \sum\{\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\| : i \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}, \lambda_{i+1} \geq \epsilon_0\} - 2\delta\bar{q}\bar{N}. \tag{3.259}$$

Let $\delta \in (0, 1)$,

$$i \in \{0, 1, \dots, \}, \ \lambda_{i+1} \geq \epsilon_0. \tag{3.260}$$

In view of (3.225) and (3.260), there exists

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \Omega_{i+1}$$

such that

$$\alpha_{i,\tau} = \lambda_{i+1} \geq \epsilon_0. \tag{3.261}$$

By (3.228) and (3.261), there exists

$$j_0 \in \{1, \dots, p(\tau) - 1\} \tag{3.262}$$

such that

$$\|y_{j_0+1}^{(i,\tau)} - y_{j_0}^{(i,\tau)}\| = \alpha_{i,\tau} \geq \epsilon_0. \tag{3.263}$$

Property (P4), (3.212), (3.214), (3.226), (3.227), and (3.263) imply that

$$\|y_{j_0}^{(i,\tau)} - P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)})\| \geq \epsilon_0,$$
$$\|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - z_{\delta,\tau_{j_0+1}}\| \leq \|y_{j_0}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| - \epsilon_1. \tag{3.264}$$

By (3.214), (3.227), and (3.264),

$$\begin{aligned}\|y_{j_0+1}^{(i,\tau)} - z_\delta\| &\le \|y_{j_0+1}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| + \delta \\ &= \|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - z_{\delta,\tau_{j_0+1}}\| + \delta \\ &\le \|y_{j_0}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| - \epsilon_1 + \delta \\ &\le \|y_{j_0}^{(i,\tau)} - z_\delta\| - \epsilon_1 + 2\delta. \end{aligned} \tag{3.265}$$

It follows from (3.17), (3.226), (3.230), (3.262), and (3.265) that

$$\begin{aligned}&\|z_\delta - x_i\| - \|z_\delta - y_{i,\tau}\| \\ &= \|z_\delta - y_0^{(i,\tau)}\| - \|z_\delta - y_{p(\tau)}^{(i,\tau)}\| \\ &= \sum_{j=0}^{p(\tau)-1} [\|z_\delta - y_j^{(i,\tau)}\| - \|z_\delta - y_{j+1}^{(i,\tau)}\|] \\ &\ge \|z_\delta - y_{j_0}^{(i,\tau)}\| - \|z_\delta - y_{j_0+1}^{(i,\tau)}\| - 2\delta(p(\tau) - 1) \\ &\ge \epsilon_1 - 2\delta\bar{q}. \end{aligned} \tag{3.266}$$

By (3.11), (3.18), (3.224), (3.232), (3.266), and the convexity of the norm,

$$\begin{aligned}&\|z_\delta - x_{i+1}\| \\ &= \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t) y_{i,t}\| \\ &\le \sum_{t\in\Omega_{i+1}} w_{i+1}(t)\|z_\delta - y_{i,t}\| \\ &= w_{i+1}(\tau)\|z_\delta - y_{i,\tau}\| \\ &+ \sum\{w_{i+1}(t)\|z_\delta - y_{i,t}\| :\ t \in \Omega_{i+1} \setminus \{\tau\}\} \\ &\le w_{i+1}(\tau)(\|z_\delta - x_i\| - \epsilon_1 + 2\delta\bar{q}) \\ &+ \sum\{w_{i+1}(t)(\|z_\delta - x_i\| + 2\delta\bar{q}) :\ t \in \Omega_{i+1} \setminus \{\tau\}\} \\ &\le \|z_\delta - x_i\| + 2\delta\bar{q} - \epsilon_1 w_{i+1}(\tau) \\ &\le \|z_\delta - x_i\| + 2\delta\bar{q} - \Delta\epsilon_1. \end{aligned}$$

Thus

$$\|z_\delta - x_{i+1}\| \le \|z_\delta - x_i\| + 2\delta\bar{q} - \Delta\epsilon_1 \text{ for all integers } i \ge 0 \text{ such that } \lambda_{i+1} \ge \epsilon_0. \tag{3.267}$$

Assume that there exists

$$i_0 \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$$

such that $\lambda_{i_0+1} \ge \epsilon_0$. In view of (3.259) and (3.267),

$$\bar{N}\bar{q}\gamma_0 \ge \Delta\epsilon_1 - 2\delta\bar{q} - 2\delta\bar{q}\bar{N}.$$

Since δ is any element of the interval (0, 1) the relation above implies that

$$\gamma_0 \ge \Delta\epsilon_1(\bar{N}\bar{q})^{-1}.$$

This contradicts (3.215). The contradiction we have reached proves

$$\lambda_{i+1} < \epsilon_0 \text{ for all } i \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}. \tag{3.268}$$

It follows from (3.225), (3.227), (3.228), and (3.268) that for every $i \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$, every $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and every $j = 0, \dots, p(t) - 1$,

$$\epsilon_0 > \lambda_{i+1} \ge \|y_{j+1}^{(i,t)} - y_j^{(i,t)}\| = \|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\|, \tag{3.269}$$

$$y_j^{(i,t)} \in F_{\epsilon_0}(P_{t_{j+1}}). \tag{3.270}$$

Let $i \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$. It follows from (3.11), (3.17), (3.224), (3.226), (3.269), and the convexity of the norm that for every $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and every $j = 0, \dots, p(t)$,

$$\|x_i - y_j^{(i,t)}\| \le \epsilon_0 p(t) \le \epsilon_0\bar{q}, \tag{3.271}$$

$$\|x_i - y_{i,t}\| \le \epsilon_0\bar{q}, \tag{3.272}$$

$$\|x_i - x_{i+1}\| = \|x_i - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t}\| \le \epsilon_0\bar{q}. \tag{3.273}$$

In view of (3.273), for each $i_1, i_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$\|x_{i_1} - x_{i_2}\| \le \bar{N}\bar{q}\epsilon_0. \tag{3.274}$$

Let $k \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$ and $s \in \{1, \dots, m\}$. By (3.218), there exist

$$j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1} \tag{3.275}$$

such that

$$s \in \{t_1, \dots, t_{p(t)}\}. \tag{3.276}$$

In view of (3.276), there exists an integer l such that

$$l \in \{0, \dots, p(t) - 1\},\ s = t_{l+1}. \tag{3.277}$$

It follows from (3.269), (3.275), and (3.277) that

$$\|y_l^{(j,t)} - P_s(y_l^{(j,t)})\| \le \epsilon_0. \tag{3.278}$$

By (3.158), (3.271), and (3.278) that

$$\begin{aligned} &\|x_j - P_s(x_j)\| \\ &\le \|x_j - y_l^{(j,t)}\| + \|y_l^{(j,t)} - P_s(y_l^{(j,t)})\| + \|P_s(y_l^{(j,t)}) - P_s(x_j)\| \\ &\le \epsilon_0 + 2\|x_j - y_l^{(j,t)}\| \le \epsilon_0(2\bar{q} + 1). \end{aligned} \tag{3.279}$$

It follows from (3.158), (3.209), (3.274), and (3.279) that

$$\begin{aligned} &\|x_k - P_s(x_k)\| \\ &\le \|x_k - x_j\| + \|x_j - P_s(x_j)\| + \|P_s(x_j) - P_s(x_k)\| \\ &\le \epsilon_0(2\bar{q} + 1) + 2\|x_k - x_j\| \\ &\le \epsilon_0(2\bar{q} + 1) + 2\epsilon_0\bar{q}\bar{N} = \epsilon_0(2\bar{q} + 1)(\bar{N} + 1) = \epsilon, \\ &x_k \in F_\epsilon(P_s),\ s = 1, \dots, m, \\ &x_k \in F_\epsilon,\ k = q\bar{N}, \dots, (q+1)\bar{N} \end{aligned}$$

and all integers $q \ge q_0$. Therefore

$$x_k \in F_\epsilon$$

for all integers $k \ge q_0\bar{N}$. In view of (3.216), $x_k \in F_\epsilon$ for all integers $k \ge Q$. Theorem 3.5 is proved.

3.9 Proof of Theorem 3.6

We may assume that $\epsilon_0 < r_0/2$. Theorem 3.6 is deduced from Theorems 2.9 and 3.5. Let $Y = X$, $\rho(y, z) = \|y - z\|$, $y, z \in X$, $\mathfrak{A}$ be the set of all mappings S defined on the set of natural numbers such that

$$S(i) = P_{\Omega_i, w_i}, \ i = 1, 2, \ldots,$$

where

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies

$$(\Omega_{i+\bar{N}}, w_{i+\bar{N}}) = (\Omega_i, w_i) \text{ for all integers } i \geq 1$$

and

$$\{1, \ldots, m\} \subset \cup_{i=1}^{\bar{N}} (\cup_{t \in \Omega_i} \{t_1, \ldots, t_{p(t)}\}).$$

Set

$$F = F_{\epsilon_0/4}.$$

Theorem 3.5 implies that property (P6) holds.

Let $Q > 0$ be as guaranteed by property (P6) and

$$\delta = 4^{-1} \epsilon_0 (Q(2\bar{N} + 1))^{-1} (\bar{q} + 1)^{-1}. \tag{3.280}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1, \ldots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \ldots, t_{p(t)}\}),$$

$$(\Omega_{i+\bar{N}}, w_{i+\bar{N}}) = (\Omega_i, w_i) \text{ for all integers } i \geq 0$$

and that sequences $\{x_i\}_{i=0}^{\infty} \subset X$, $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy

$$x_0 \in B(0, M),$$

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta) \text{ for all integers } i \geq 1.$$

Let $i \geq 0$ be an integer. The inclusion

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), \delta)$$

holds. By (3.20), there exist vectors

$$(y_{i,t}, \alpha_{i,t}) \in A_0(x_i, t, \delta), \ t \in \Omega_{i+1} \tag{3.281}$$

such that

$$\|x_{i+1} - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\| \leq \delta, \tag{3.282}$$

$$\lambda_{i+1} = \max\{\alpha_{i,t} : \ t \in \Omega_{i+1}\}.$$

It follows from (3.19) and the relation above that for every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(i,t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(i,t)} = x_i, \ y_{p(t)}^{(i,t)} = y_{i,t}, \tag{3.283}$$

for every integer $j = 1, \dots, p(t)$,

$$\|y_j^{(i,t)} - P_{t_j}(y_{j-1}^{(i,t)})\| \leq \delta, \tag{3.284}$$

$$\alpha_{i,t} = \max\{\|y_{j+1}^{(i,t)} - y_j^{(i,t)}\| : \ j = 0, \dots, p(t) - 1\}. \tag{3.285}$$

Proposition 2.8, (3.17), (3.158), (3.283), and (3.284) imply that for every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$,

$$\|y_{i,t} - P[t](x_i)\| \leq \|y_{p(t)}^{(i,t)} - P[t](x_i)\| \leq p(t)\delta \leq \bar{q}\delta.$$

By the relation above, (3.11), (3.12), (3.280), (3.282), and the convexity of the norm,

$$\begin{aligned} &\|x_{i+1} - P_{\Omega_{i+1}, w_{i+1}}(x_i)\| \\ &\leq \|x_{i+1} - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\| + \|\sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t} - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) P[t](x_i)\| \\ &\leq (\bar{q} + 1)\delta \leq 4^{-1}\epsilon_0 (Q(2\bar{N} + 1))^{-1}. \end{aligned} \tag{3.286}$$

Theorem 2.9, the choice of Q and (3.286) imply that for all integers $i \geq Q$,

$$B(x_i, \epsilon_0/4) \cap F_{\epsilon_0/4} \neq \emptyset. \tag{3.287}$$

Let $i \geq Q$ be an integer. In view of (3.287), there exists

$$\xi \in B(x_i, \epsilon_0/4) \cap F_{\epsilon_0/4}. \tag{3.288}$$

Then by (3.158) and (3.288), for all $s = 1, \dots, m$,

$$\begin{aligned} &\|x_i - P_s(x_i)\| \\ &\leq \|x_i - \xi\| + \|\xi - P_s(\xi)\| + \|P_s(\xi) - P_s(x_i)\| \\ &\leq 2\|x_i - \xi\| + \epsilon_0/4 < \epsilon_0. \end{aligned}$$

Theorem 3.6 is proved.

3.10 The Third Problem

Recall that $M_* > 1$. For $i = 1, \dots, m$, set

$$C_i = \mathrm{Fix}(P_i). \tag{3.289}$$

We suppose that the following assumption holds.

(A3) For each $M > 0$ and each $\gamma > 0$ there exists $\delta > 0$ such that for each $i \in \{1, \dots, m\}$, each $x \in B(0, M)$ satisfying $d(x, C_i) \geq \gamma$ and each

$$z \in B(0, M) \cap C_i$$

the inequality

$$\|P_i(x) - z\| \leq \|x - z\| - \delta$$

is true.

In this chapter we prove the following three results: Theorem 3.7 which shows that the inexact dynamic string-averaging method generates approximate solutions if perturbations are summable, Theorem 3.8 which establishes that the exact dynamic string-averaging method generates approximate solutions, and Theorem 3.9 which demonstrates that the inexact dynamic string-averaging method generates approximate solutions if the perturbations are small enough.

Theorem 3.7 *Let*

$$M \geq M_*, \ \epsilon \in (0, 1)$$

and let a sequence $\{\epsilon_i\}_{i=1}^{\infty} \subset (0, \infty)$ *satisfy*

$$\Lambda := \sum_{i=1}^{\infty} \epsilon_i < \infty. \tag{3.290}$$

Then there exists a number $Q > 0$ *such that for each*

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

which satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\})$$

and each pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$ *and* $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *which satisfies*

$$x_0 \in B(0, M)$$

and

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \epsilon_i), \ i = 1, 2, \dots,$$

the following inequality holds:

$$Card(\{i \in \{0, 1, \dots\} : \ \max\{d(x_i, C_s) : \ s = 1, \dots, m\} > \epsilon\}) \leq Q.$$

Theorem 3.8 *Assume that for each* $x, y \in X$ *and each* $i \in \{1, \dots, m\}$,

$$\|P_i(x) - P_i(y)\| \leq \|x - y\|. \tag{3.291}$$

Let $M \geq M_*$, $\epsilon \in (0, 1)$. *Then there exists a constant* $Q > 0$ *such that for each*

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfying for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\})$$

and

$$(\Omega_{i+\bar{N}}, w_{i+\bar{N}}) = (\Omega_i, w_i) \text{ for all integers } i \geq 0$$

and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ *which satisfies*

$$x_0 \in B(0, M),$$

$$x_{i+1} = P_{\Omega_{i+1}, w_{i+1}}(x_i)$$

for each integer $i \ge 0$, the inequality

$$d(x_i, C_s) \le \epsilon, \ s = 1, \dots, m$$

holds for all integers $i \ge Q$.

Theorem 3.9 *Assume that for each $x, y \in X$ and each $i \in \{1, \dots, m\}$,*

$$\|P_i(x) - P_i(y)\| \le \|x - y\|.$$

Let $M \ge M_*$, $r_0 \in (0, 1)$,

$$\{x \in X : \ d(x, C_s) \ \le r_0, \ s = 1, \dots, m\} \subset B(0, M),$$

$\epsilon \in (0, 1)$. *Then there exists $Q > 0$, $\delta > 0$ such that for each*

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfying for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\})$$

and

$$(\Omega_{i+\bar{N}}, w_{i+\bar{N}}) = (\Omega_i, w_i) \text{ for all integers } i \ge 0$$

and each pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$ and $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ which satisfies

$$x_0 \in B(0, M),$$
$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta)$$

for all natural numbers i, the inequality

$$d(x_i, C_s) \le \epsilon, \ s = 1, \dots, m$$

holds for all integers $i \ge Q$.

3.11 Proof of Theorem 3.7

By (A1), for every $\delta > 0$ there exists

$$z_\delta \in B(0, M_*) \tag{3.292}$$

such that

$$B(z_\delta, \delta) \cap C_i = B(z_\delta, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset, \ i = 1, \dots, m. \tag{3.293}$$

In view of (3.293), for each $\delta > 0$ and each $i \in \{1, \dots, m\}$ there exists

$$z_{\delta,i} \in B(z_\delta, \delta) \cap C_i. \tag{3.294}$$

Set

$$\gamma_0 = \epsilon(\bar{N} + 1)^{-1}(3\bar{q} + 1)^{-1}. \tag{3.295}$$

By (A3), there exists $\gamma \in (0, \gamma_0)$ such that the following property holds:

(P6) for each $i \in \{1, \dots, m\}$, each

$$z \in B(0, 3M + 1 + \Lambda) \cap C_i$$

and each $x \in B(0, 3M + 1 + \Lambda(\bar{q} + 1))$ satisfying $d(x, C_i) \geq \gamma_0/2$, the inequality

$$\|P_i(x) - z\| \leq \|x - z\| - \gamma$$

is true.

In view of (3.290), there exists a natural number n_0 such that

$$\epsilon_i < \gamma/4 \text{ for all integers } i \geq n_0. \tag{3.296}$$

Set

$$Q = n_0 + 8\bar{N}(\Delta\gamma)^{-1}(M + (\bar{q} + 1)\Lambda). \tag{3.297}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{3.298}$$

satisfies for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}), \tag{3.299}$$

$\{x_i\}_{i=0}^{\infty} \subset X$ and $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfies

$$x_0 \in B(0, M) \tag{3.300}$$

and

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \epsilon_i), \ i = 1, 2, \dots, \tag{3.301}$$

Let $i \geq 0$ be an integer. By (3.301),

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), \epsilon_{i+1}). \tag{3.302}$$

By (3.20) and (3.302), there exist vectors

$$(y_{i,t}, \alpha_{i,t}) \in A_0(x_i, t, \epsilon_{i+1}), \ t \in \Omega_{i+1} \tag{3.303}$$

such that

$$\|x_{i+1} - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\| \leq \epsilon_{i+1}, \tag{3.304}$$

$$\lambda_{i+1} = \max\{\alpha_{i,t} : \ t \in \Omega_{i+1}\}. \tag{3.305}$$

It follows from (3.19) and (3.303) that for every index vector

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$$

there exists a finite sequence $\{y_j^{(i,t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(i,t)} = x_i, \ y_{p(t)}^{(i,t)} = y_{i,t}, \tag{3.306}$$

for every integer $j = 1, \dots, p(t)$,

$$\|y_j^{(i,t)} - P_{t_j}(y_{j-1}^{(i,t)})\| \leq \epsilon_{i+1}, \tag{3.307}$$

$$\alpha_{i,t} = \max\{\|y_{j+1}^{(i,t)} - y_j^{(i,t)}\| : \ j = 0, \dots, p(t) - 1\}. \tag{3.308}$$

For every $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$, set

$$\beta_{i,t} = \max\{d(y_j^{(i,t)}, C_{t_{j+1}}) : \ j = 0, \dots, p(t) - 1\}. \tag{3.309}$$

$$\mu_{i+1} = \max\{\beta_{i,t} : \ t \in \Omega_{i+1}\}. \tag{3.310}$$

Set

$$\epsilon_0 = 0. \tag{3.311}$$

Let $\delta > 0$. By (3.292) and (3.300),

$$\|z_\delta - x_0\| \le 2M. \tag{3.312}$$

Let $i \ge 0$ be an integer,

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1},\ \ j = 0, \ldots, p(t) - 1. \tag{3.313}$$

Relations (3.7), (3.294), (3.307), and (3.313) imply that

$$\begin{aligned}
\|z_\delta - y_{j+1}^{(i,t)}\| &\le \|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\| + \|P_{t_{j+1}}(y_j^{(i,t)}) - y_{j+1}^{(i,t)}\| \\
&\le \|z_\delta - z_{\delta,t_{j+1}}\| + \|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\| + \epsilon_{i+1} \\
&\le \delta + \epsilon_{i+1} + \|z_{\delta,t_{j+1}} - y_j^{(i,t)}\| \\
&\le \|z_\delta - y_j^{(i,t)}\| + 2\delta + \epsilon_{i+1}
\end{aligned}$$

and

$$\|z_\delta - y_{j+1}^{(i,t)}\| \le \|z_\delta - y_j^{(i,t)}\| + 2\delta + \epsilon_{i+1}. \tag{3.314}$$

By (3.17), (3.306), and (3.314), for all $j = 0, \ldots, p(t)$,

$$\begin{aligned}
\|z_\delta - y_j^{(i,t)}\| &\le \|z_\delta - y_0^{(i,t)}\| + j(2\delta + \epsilon_{i+1}) \\
&= \|z_\delta - x_i\| + j(2\delta + \epsilon_{i+1}) \\
&\le \|z_\delta - x_i\| + \bar{q}(2\delta + \epsilon_{i+1}).
\end{aligned} \tag{3.315}$$

By (3.306) and (3.315),

$$\|z_\delta - y_{i,t}\| = \|z_\delta - y_{p(t)}^{(i,t)}\| \le \|z_\delta - x_i\| + \bar{q}(2\delta + \epsilon_{i+1}). \tag{3.316}$$

By (3.11), (3.304), and the convexity of the norm,

$$\begin{aligned}
\|z_\delta - x_{i+1}\| &\le \|z_\delta - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\| + \|\sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t} - x_{i+1}\| \\
&\le \sum_{t \in \Omega_{i+1}} w_{i+1}(t)\|z_\delta - y_{i,t}\| + \epsilon_{i+1} \\
&\le \|z_\delta - x_i\| + (\bar{q} + 1)(2\delta + \epsilon_{i+1}).
\end{aligned} \tag{3.317}$$

By induction we show that for all integers $i \geq 0$,

$$\|z_\delta - x_i\| \leq 2M + 2(\bar{q}+1)\delta i + (\sum_{j=0}^{i} \epsilon_j)(\bar{q}+1). \tag{3.318}$$

In view of (3.311) and (3.312), inequality (3.318) is true for $i = 0$.

Assume that $i \geq 0$ is an integer and that (3.318) holds. By (3.317) and (3.318),

$$\|z_\delta - x_{i+1}\| \leq \|z_\delta - x_i\| + (\bar{q}+1)(2\delta + \epsilon_{i+1})$$
$$\leq 2M + 2(\bar{q}+1)\delta(i+1) + (\sum_{j=0}^{i+1} \epsilon_j)(\bar{q}+1).$$

Therefore by induction we showed that (3.318) holds for all integers $i \geq 0$. It follows from (3.315) and (3.318) that for all integers $i \geq 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t)$,

$$\|z_\delta - y_j^{(i,t)}\| \leq \|z_\delta - x_i\| + \bar{q}(2\delta + \epsilon_{i+1})$$
$$\leq 2M + 2(\bar{q}+1)\delta(i+1) + (\sum_{j=0}^{i+1} \epsilon_j)(\bar{q}+1). \tag{3.319}$$

Let n be a natural number. By (3.290), (3.292), and (3.318), for all integers $i = 0, \dots, n$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t)$,

$$\|x_i\| \leq \|z_\delta\| + \|x_i - z_\delta\| \leq 3M + 2(\bar{q}+1)\delta n + \Lambda(\bar{q}+1),$$
$$\|y_j^{(i,t)}\| \leq \|z_\delta\| + \|y_j^{(i,t)} - z_\delta\| \leq 3M + 2(\bar{q}+1)\delta n + \Lambda(\bar{q}+1).$$

Since the relation above holds for any $\delta > 0$ we conclude that for all integers $i \geq 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t)$,

$$\|x_i\| \leq 3M + \Lambda(\bar{q}+1), \tag{3.320}$$
$$\|y_j^{(i,t)}\| \leq 3M + +\Lambda(\bar{q}+1). \tag{3.321}$$

Set

$$E_0 = \{i \in \{n_0, n_0+1, \dots\} : \ \mu_{i+1} \geq \gamma_0\}, \tag{3.322}$$
$$E_1 = \{n_0, n_0+1, \dots, \} \setminus E_0. \tag{3.323}$$

Let

$$i \in E_0.$$

By (3.310) and (3.322),

$$\mu_{i+1} \geq \gamma_0$$

and there exists

$$\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \Omega_{i+1}$$

such that

$$\beta_{i,\tau} = \mu_{i+1} \geq \gamma_0. \tag{3.324}$$

By (3.309) and (3.324), there exists

$$j_0 \in \{1, \ldots, p(\tau) - 1\} \tag{3.325}$$

such that

$$d(y_{j_0}^{(i,\tau)}, C_{\tau_{j_0+1}}) = \beta_{i,\tau} \geq \gamma_0. \tag{3.326}$$

Property (P6), (3.292), (3.293), (3.321), (3.325), and (3.326) imply that

$$\|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - z_{\delta,\tau_{j_0+1}}\| \leq \|y_{j_0}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| - \gamma. \tag{3.327}$$

By (3.294), (3.296), (3.307), (3.322), and (3.325),

$$\begin{aligned}
\|y_{j_0+1}^{(i,\tau)} - z_\delta\| &\leq \|y_{j_0+1}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| + \|z_{\delta,\tau_{j_0+1}} - z_\delta\| \\
&\leq \|y_{j_0+1}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| + \delta \\
&\leq \|y_{j_0+1}^{(i,\tau)} - P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)})\| + \|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - z_{\delta,\tau_{j_0+1}}\| + \delta \\
&\leq \epsilon_{i+1} + \|y_{j_0}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| - \gamma + \delta \\
&\leq \gamma/4 - \gamma + \delta + \|y_{j_0}^{(i,\tau)} - z_\delta\| + \delta \\
&\leq 2\delta - 3\gamma/4 + \|y_{j_0}^{(i,\tau)} - z_\delta\|,
\end{aligned}$$

$$\|y_{j_0+1}^{(i,\tau)} - z_\delta\| \leq \|y_{j_0}^{(i,\tau)} - z_\delta\| - 3\gamma/4 + 2\delta. \tag{3.328}$$

By (3.17), (3.306), (3.314), (3.325), and (3.328),

$$\begin{aligned}
&\|z_\delta - x_i\| - \|z_\delta - y_{i,\tau}\| \\
&= \sum_{j=0}^{p(\tau)-1} [\|z_\delta - y_j^{(i,\tau)}\| - \|z_\delta - y_{j+1}^{(i,\tau)}\|] \\
&\geq \|y_{j_0}^{(i,\tau)} - z_\delta\| - \|y_{j_0+1}^{(i,\tau)} - z_\delta\| - (p(\tau)-1)(2\delta + \epsilon_{i+1}) \\
&\geq 3\gamma/4 - 2\delta - (\bar{q}-1)(2\delta + \epsilon_{i+1}). \qquad (3.329)
\end{aligned}$$

It follows from (3.11), (3.18), (3.304), (3.316), and the convexity of the function $\|\cdot\|$ that

$$\begin{aligned}
&\|z_\delta - x_{i+1}\| \\
&\leq \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t) y_{i,t}\| + \|\sum_{t\in\Omega_{i+1}} w_{i+1}(t) y_{i,t} - x_{i+1}\| \\
&\leq \sum_{t\in\Omega_{i+1}} w_{i+1}(t)\|z_\delta - y_{i,t}\| + \epsilon_{i+1} \\
&\leq \epsilon_{i+1} + \|z_\delta - x_i\| + \sum_{t\in\Omega_{i+1}} w_{i+1}(t)[\|z_\delta - y_{i,t}\| - \|z_\delta - x_i\|] \\
&\leq \epsilon_{i+1} + \|z_\delta - x_i\| + w_{i+1}(\tau)[\|z_\delta - y_{i,\tau}\| - \|z_\delta - x_i\|] \\
&+ \sum\{w_{i+1}(t)[\|z_\delta - y_{i,t}\| - \|z_\delta - x_i\|] : \ t \in \Omega_{i+1} \setminus \{\tau\}\} \\
&\leq \epsilon_{i+1} + \|z_\delta - x_i\| + w_{i+1}(\tau)(-3\gamma/4 + 2\delta + (\bar{q}-1)(2\delta + \epsilon_{i+1})) + \bar{q}(2\delta + \epsilon_{i+1}), \\
&\|z_\delta - x_{i+1}\| \leq \|z_\delta - x_i\| - 3\Delta\gamma/4 + 2\bar{q}(2\delta + \epsilon_{i+1}) \qquad (3.330)
\end{aligned}$$

for all $i \in E_0$. By (3.292), (3.317), (3.320), (3.322), (3.323), and (3.330), for every integer $n > n_0$,

$$\begin{aligned}
&4M + \Lambda(\bar{q}+1) \geq \|z_\delta - x_{n_0}\| \\
&\geq \|z_\delta - x_{n_0}\| - \|z_\delta - x_n\| \\
&\geq \sum_{i=n_0}^{n-1} (\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\|) \\
&= \sum_{i\in E_0\cap[0,n-1]} (\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\|) \\
&+ \sum_{i\in E_1\cap[0,n-1]} (\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\|)
\end{aligned}$$

$$\geq \mathrm{Card}(E_0 \cap [0, n-1])(3\Delta\gamma/4) - 4\bar{q}\delta n - 2\bar{q}\sum_{i=0}^{n}\epsilon_i - (\bar{q}+1)\sum_{i \in E_1 \cap [0,n-1]}(2\delta + \epsilon_{i+1})$$

$$\geq \mathrm{Card}(E_0 \cap [0, n-1])(3\Delta\gamma/4) - \Lambda(3\bar{q}+1) - 4\bar{q}\delta n - 2(\bar{q}+1)\delta n.$$

Since δ is any element of the interval (0, 1) we conclude that

$$(3\Delta\gamma/4)\mathrm{Card}(E_0 \cap [0, n-1]) \leq 4M + 4\Lambda(\bar{q}+1),$$

$$\mathrm{Card}(E_0 \cap [0, n-1]) \leq 2\Delta^{-1}\gamma^{-1}(4M + 4\Lambda(\bar{q}+1)).$$

Since n is any natural number satisfying $n > n_0$ we conclude that

$$\mathrm{Card}(E_0) \leq 2\Delta^{-1}\gamma^{-1}(4M + 4\Lambda(\bar{q}+1)). \tag{3.331}$$

Set

$$E_2 = \{i \in \{n_0, n_0+1, \dots\} : \ [i, i+\bar{N}-1] \cap E_0 \neq \emptyset\}. \tag{3.332}$$

By (3.331) and (3.332),

$$\mathrm{Card}(E_2) \leq \bar{N}\mathrm{Card}(E_0)$$

$$\leq 8\bar{N}\Delta^{-1}\gamma^{-1}(M + (\bar{q}+1)\Lambda)). \tag{3.333}$$

Let an integer $j \geq n_0$ satisfy

$$j \notin E_2. \tag{3.334}$$

In view of (3.332) and (3.334),

$$[j, j+\bar{N}-1] \cap E_0 = \emptyset$$

and for all $i = j, \dots, j+\bar{N}-1$,

$$\mu_{i+1} < \gamma_0. \tag{3.335}$$

By (3.7), (3.17), (3.296), (3.306), (3.309), (3.310), and (3.335), for all $i = j, \dots, j+\bar{N}-1$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $s = 0, \dots, p(t)-1$,

$$\gamma_0 > \mu_{i+1} \geq \beta_{i,t} \geq d(y_s^{(i,t)}, C_{t_{s+1}}) \tag{3.336}$$

and there exists

$$\xi \in C_{t_{s+1}}$$

such that

$$\|y_s^{(i,t)} - \xi\| < \gamma_0,$$

$$\|y_s^{(i,t)} - P_{t_{s+1}}(y_s^{(i,t)})\|$$

$$\leq \|y_s^{(i,t)} - \xi\| + \|\xi - P_{t_{s+1}}(y_s^{(i,t)})\|$$

$$\leq 2\|y_s^{(i,t)} - \xi\| < 2\gamma_0$$

and

$$\|y_s^{(i,t)} - y_{s+1}^{(i,t)}\|$$

$$\leq \|y_s^{(i,t)} - P_{t_{s+1}}(y_s^{(i,t)})\| + \|P_{t_{s+1}}(y_s^{(i,t)}) - y_{s+1}^{(i,t)}\|$$

$$\leq 2\gamma_0 + \epsilon_{i+1} < 2\gamma_0 + \gamma_0/4. \tag{3.337}$$

Relations (3.336) and (3.337) imply that for $i = j, \dots, j + \bar{N} - 1$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $s = 0, \dots, p(t) - 1$,

$$\|x_i - y_s^{(i,t)}\| \leq \bar{q}\gamma_0(2 + 1/4),$$

$$d(x_i, C_{t_{s+1}}) \leq \|x_i - y_s^{(i,t)}\| + d(y_s^{(i,t)}, C_{t_{s+1}})$$

$$\leq \bar{q}\gamma_0(2 + 1/4) + \gamma_0,\ \ s = 0, \dots, p(t) - 1. \tag{3.338}$$

By (3.306) and (3.337), for each $i = j, \dots, j + \bar{N} - 1$ and each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$,

$$\|x_i - y_{i,t}\| \leq \bar{q}\gamma_0(2 + 1/4). \tag{3.339}$$

It follows from (3.11), (3.296), (3.304), (3.339), and the convexity of the norm that for all $i = j, \dots, j + \bar{N} - 1$

$$\|x_{i+1} - x_i\|$$

$$\leq \|x_i - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\| + \|\sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t} - x_{i+1}\|$$

$$\leq \epsilon_{i+1} + \sum_{t \in \Omega_{i+1}} w_{i+1}(t)\|y_{i,t} - x_i\|$$

$$\leq \epsilon_{i+1} + \gamma_0\bar{q}(2 + 1/4) \leq \gamma_0\bar{q}(2 + 1/4) + \gamma_0/4. \tag{3.340}$$

By (3.340), for each pair of integers $i_1, i_2 \in \{j, \dots, j + \bar{N}\}$,

$$\|x_{i_1} - x_{i_2}\| \leq \bar{q}\bar{N}\gamma_0(2 + 1/4) + \bar{N}\gamma_0/4. \tag{3.341}$$

Let $s \in \{1, \dots, m\}$. In view of (3.299), there exist

$$i \in \{j, \dots, j + \bar{N} - 1\}, \; t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$$

such that

$$s \in \{t_1, \dots, t_{p(t)}\}.$$

In view of (3.338),

$$d(x_i, C_s) \le \bar{q}\gamma_0(2 + 1/4) + \gamma_0. \tag{3.342}$$

By (3.295), (3.341), and (3.342),

$$d(x_j, C_s) \le \|x_j - x_i\| + d(x_i, C_s)$$

$$\le \bar{N}\bar{q}\gamma_0(2 + 1/4) + \bar{N}\gamma_0/4 + \gamma_0(2\bar{q} + \bar{q}/4 + 1) \le 3(\bar{N} + 1)\gamma_0\bar{q} \le \epsilon,$$

$$d(x_j, C_s) \le \epsilon, \; s = 1, \dots, m \tag{3.343}$$

for every $j \in \{n_0, n_0 + 1, \dots\}$ such that $j \notin E_2$. By (3.333) and (3.343),

$$\mathrm{Card}(\{j \in \{0, 1, \dots\} : \; \max\{d(x_j, C_s) : \; s = 1, \dots, m\} > \epsilon\})$$

$$\le n_0 + \mathrm{Card}(E_2)$$

$$\le n_0 + 2\Delta^{-1}\gamma^{-1}\bar{N}(4M + 4\Lambda(\bar{q} + 1)) = Q.$$

Theorem 3.7 is proved.

3.12 Proof of Theorem 3.8

By (A1), for every $\delta > 0$ there exists

$$z_\delta \in B(0, M_*) \tag{3.344}$$

such that

$$B(z_\delta, \delta) \cap C_i \ne \emptyset, \; i = 1, \dots, m. \tag{3.345}$$

In view of (3.345), for each $\delta > 0$ and each $i \in \{1, \dots, m\}$ there exists

$$z_{\delta,i} \in B(z_\delta, \delta) \cap C_i. \tag{3.346}$$

Set

$$\epsilon_0 = \epsilon(\bar{N} + 1)^{-1}(2\bar{q} + 1)^{-1}. \tag{3.347}$$

By (3.8), (3.11), (3.12), (3.291), and convexity of the norm, for each $(\Omega, w) \in \mathcal{M}_*$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega$ and all $x, y \in X$,

$$\|P[t](x) - P[t](y)\| \leq \|x - y\|, \tag{3.348}$$

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\|$$

$$= \|\sum_{t\in\Omega} w(t)P[t](x) - \sum_{t\in\Omega} w(t)P[t](y)\| \leq \|x - y\|. \tag{3.349}$$

By (A3), there exists $\epsilon_1 \in (0, \epsilon_0)$ such that the following property holds:

(P7) for each $i \in \{1, \dots, m\}$, each

$$z \in B(0, 3M + 1) \cap C_i$$

and each $x \in B(0, 3M + 1)$ satisfying $d(x, C_i) \geq \epsilon_0$, the inequality

$$\|P_i(x) - z\| \leq \|x - z\| - \epsilon_1$$

is true.

Set

$$\gamma_0 = \epsilon_1(2\bar{N} + 1)^{-1}(8\bar{q})^{-1}\Delta. \tag{3.350}$$

By (A3), there exists $\gamma \in (0, \gamma_0)$ such that the following property holds:

(P8) for each $i \in \{1, \dots, m\}$, each

$$z \in B(0, 3M + 1) \cap C_i$$

and each $x \in B(0, 3M + 1)$ satisfying $d(x, C_i) \geq \gamma_0/2$, the inequality

$$\|P_i(x) - z\| \leq \|x - z\| - \gamma$$

is true.

Set

$$Q = \bar{N}((\Delta\gamma)^{-1}2M + 1). \tag{3.351}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{3.352}$$

satisfies for each natural number j,

$$\{1, \ldots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \ldots, t_{p(t)}\}) \tag{3.353}$$

and

$$(\Omega_{i+\bar{N}}, w_{i+\bar{N}}) = (\Omega_i, w_i) \text{ for all integers } i \geq 0 \tag{3.354}$$

and $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies

$$x_0 \in B(0, M) \tag{3.355}$$

and

$$x_{i+1} = P_{\Omega_{i+1}, w_{i+1}}(x_i) \text{ for all integers } i \geq 0. \tag{3.356}$$

Set

$$T = \prod_{i=1}^{\bar{N}} P_{\Omega_i, w_i} = P_{\Omega_{\bar{N}}, w_{\bar{N}}} \cdots P_{\Omega_1, w_1}. \tag{3.357}$$

Let $i \geq 0$ be an integer. By (3.8), (3.12), (3.19), (3.20), and (3.356), there exists $\lambda_{i+1} \geq 0$ such that

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), 0). \tag{3.358}$$

By (3.20) and (3.358), there exist vectors

$$(y_{i,t}, \alpha_{i,t}) \in A_0(x_i, t, 0), \;\; t \in \Omega_{i+1} \tag{3.359}$$

such that

$$x_{i+1} = \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}, \tag{3.360}$$

$$\lambda_{i+1} = \max\{\alpha_{i,t} \, : \; t \in \Omega_{i+1}\}. \tag{3.361}$$

It follows from (3.19) and (3.359) that for every index vector

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$$

there exists a finite sequence $\{y_j^{(i,t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(i,t)} = x_i, \; y_{p(t)}^{(i,t)} = y_{i,t}, \tag{3.362}$$

for every integer $j = 1, \ldots, p(t)$,

$$y_j^{(i,t)} = P_{t_j}(y_{j-1}^{(i,t)}), \tag{3.363}$$

$$\alpha_{i,t} = \max\{\|y_{j+1}^{(i,t)} - y_j^{(i,t)}\| : \; j = 0, \ldots, p(t) - 1\}. \tag{3.364}$$

For every $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ set

$$\beta_{i,t} = \max\{d(y_j^{(i,t)}, C_{t_{j+1}}) : \; j = 0, \ldots, p(t) - 1\}, \tag{3.365}$$

$$\mu_{i+1} = \max\{\beta_{i,t} : \; t \in \Omega_{i+1}\}. \tag{3.366}$$

Let $\delta > 0$. In view of (3.344) and (3.355),

$$\|z_\delta - x_0\| \le 2M. \tag{3.367}$$

Let $i \ge 0$ be an integer,

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}, \; j = 0, \ldots, p(t) - 1.$$

Relations (3.7), (3.346), and (3.363) imply that

$$\begin{aligned} \|z_\delta - y_{j+1}^{(i,t)}\| &\le \|z_\delta - P_{t_{j+1}}(y_j^{(i,t)})\| \\ &\le \|z_\delta - z_{\delta,t_{j+1}}\| + \|z_{\delta,t_{j+1}} - P_{t_{j+1}}(y_j^{(i,t)})\| \\ &\le \delta + \|z_{\delta,t_{j+1}} - y_j^{(i,t)}\| \\ &\le \|z_\delta - y_j^{(i,t)}\| + 2\delta. \end{aligned} \tag{3.368}$$

By (3.17), (3.362), and (3.368), for all $j = 0, \ldots, p(t)$,

$$\begin{aligned} \|z_\delta - y_j^{(i,t)}\| &\le \|z_\delta - y_0^{(i,t)}\| + 2\delta j \\ &= \|z_\delta - x_i\| + 2\delta j \\ &\le \|z_\delta - x_i\| + 2\bar{q}\delta. \end{aligned} \tag{3.369}$$

By (3.362) and (3.369),

$$\|z_\delta - y_{i,t}\| = \|z_\delta - y_{p(t)}^{(i,t)}\| \le \|z_\delta - x_i\| + 2\delta\bar{q}. \tag{3.370}$$

By (3.11), (3.360), (3.370), and the convexity of the norm,

$$\|z_\delta - x_{i+1}\| \le \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t) y_{i,t}\|$$
$$\le \sum_{t\in\Omega_{i+1}} w_{i+1}(t)\|z_\delta - y_{i,t}\|$$
$$\le \|z_\delta - x_i\| + 2\delta\bar{q}. \tag{3.371}$$

By induction, using (3.367) and (3.371), we can show that for all integers $i \ge 0$,

$$\|z_\delta - x_i\| \le \|z_\delta - x_0\| + 2\bar{q}\delta i \le 2M + 2\bar{q}\delta i. \tag{3.372}$$

It follows from (3.369) and (3.372) that for all integers $i \ge 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t)$,

$$\|z_\delta - y_j^{(i,t)}\| \le \|z_\delta - x_i\| + 2\bar{q}\delta$$
$$\le \|z_\delta - x_0\| + 2\bar{q}\delta(i+1)$$
$$\le 2M + 2\bar{q}\delta(i+1). \tag{3.373}$$

By (3.344), (3.372), and (3.373), for all integers $i \ge 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t)$,

$$\|y_j^{(i,t)}\| \le \|z_\delta\| + \|y_j^{(i,t)} - z_\delta\| \le 3M + 2\bar{q}\delta(i+1),$$
$$\|x_i\| \le \|z_\delta\| + \|x_i - z_\delta\| \le 3M + 2\bar{q}\delta i.$$

Since the relations above hold for any $\delta > 0$ we conclude that for all integers $i \ge 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t)$,

$$\|x_i\| \le 3M, \ \|y_j^{(i,t)}\| \le 3M. \tag{3.374}$$

Set

$$E_0 = \{i \in \{0, 1, \dots\} : \ \mu_{i+1} \ge \gamma_0\}, \tag{3.375}$$
$$E_1 = \{0, 1, \dots,\} \setminus E_0. \tag{3.376}$$

Let

$$i \in E_0, \ \delta \in (0, 1).$$

By (3.366) and (3.375),

$$\mu_{i+1} \geq \gamma_0$$

and there exists

$$\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \Omega_{i+1}$$

such that

$$\beta_{i,\tau} = \mu_{i+1} \geq \gamma_0. \tag{3.377}$$

By (3.365) and (3.377), there exists

$$j_0 \in \{1, \ldots, p(\tau) - 1\}$$

such that

$$d(y_{j_0}^{(i,\tau)}, C_{\tau_{j_0+1}}) = \beta_{i,\tau} \geq \gamma_0. \tag{3.378}$$

Property (P8), (3.344), (3.346), (3.374), and (3.378) that

$$\|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - z_{\delta,\tau_{j_0+1}}\| \leq \|y_{j_0}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| - \gamma. \tag{3.379}$$

By (3.346), (3.363), and (3.379),

$$\begin{aligned}
\|y_{j_0+1}^{(i,\tau)} - z_\delta\| &= \|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - z_\delta\| \\
&\leq \|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - z_{\delta,\tau_{j_0+1}}\| + \|z_{\delta,\tau_{j_0+1}} - z_\delta\| \\
&\leq \|y_{j_0}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| - \gamma + \delta \\
&\leq \|y_{j_0}^{(i,\tau)} - z_\delta\| - \gamma + 2\delta.
\end{aligned} \tag{3.380}$$

By (3.17), (3.362), (3.368), and (3.380),

$$\begin{aligned}
&\|z_\delta - x_i\| - \|z_\delta - y_{i,\tau}\| \\
&= \sum_{j=0}^{p(\tau)-1} [\|z_\delta - y_j^{(i,\tau)}\| - \|z_\delta - y_{j+1}^{(i,\tau)}\|] \\
&\geq \|y_{j_0}^{(i,\tau)} - z_\delta\| - \|y_{j_0+1}^{(i,\tau)} - z_\delta\| - 2\delta(p(\tau) - 1) \\
&\geq \gamma - 2\delta\bar{q}.
\end{aligned} \tag{3.381}$$

It follows from (3.11) and the convexity of the function $\|\cdot\|$ that

$$\|z_\delta - x_{i+1}\|$$
$$= \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_{i,t}\|$$
$$\leq \sum_{t\in\Omega_{i+1}} w_{i+1}(t)\|z_\delta - y_{i,t}\|. \tag{3.382}$$

By (3.11), (3.18), (3.370), (3.381), and (3.382),

$$\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\|$$
$$\geq \sum_{t\in\Omega_{i+1}} w_{i+1}(t)[\|z_\delta - x_i\| - \|z_\delta - y_{i,t}\|]$$
$$= w_{i+1}(\tau)[\|z_\delta - x_i\| - \|z_\delta - y_{i,\tau}\|]$$
$$+\sum\{w_{i+1}(t)[\|z_\delta - x_i\| - \|z_\delta - y_{i,t}\|] :\ t \in \Omega_{i+1} \setminus \{\tau\}\}$$
$$\geq w_{i+1}(\tau)(\gamma - 2\delta\bar{q}) \sum\{w_{i+1}(t)(2\delta\bar{q}) :\ t \in \Omega_{i+1} \setminus \{\tau\}\}$$
$$\geq \Delta\gamma - 2\delta\bar{q},$$
$$\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\| \geq \Delta\gamma - 2\delta\bar{q} \text{ for all } i \in E_0. \tag{3.383}$$

Set

$$E_2 = \{k \in \{0, 1, \dots\} :\ \max\{\mu_{i+1} :\ i = k\bar{N}, \dots, (k+1)\bar{N}-1\} \geq \gamma_0\}. \tag{3.384}$$

Let $\delta \in (0, 1)$ and n be a natural number. By (3.367),

$$2M \geq \|z_\delta - x_0\|$$
$$\geq \|z_\delta - x_0\| - \|z_\delta - x_{\bar{N}n}\|$$
$$= \sum_{k=0}^{n-1}(\|z_\delta - x_{k\bar{N}}\| - \|z_\delta - x_{(k+1)\bar{N}}\|)$$
$$= \sum_{k=0}^{n-1}(\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (\|z_\delta - x_j\| - \|z_\delta - x_{j+1}\|)). \tag{3.385}$$

Assume that an integer $k \in [0, n-1]$ satisfies

$$k \in E_2.$$

By (3.371), (3.376), (3.381), and (3.384),

$$\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (\|z_\delta - x_j\| - \|z_\delta - x_{j+1}\|)$$

$$= \sum \{\|z_\delta - x_j\| - \|z_\delta - x_{j+1}\| : \ j \in \{k\bar{N}, \dots, (k+1)\bar{N} - 1\}, \ \mu_{j+1} \geq \gamma_0\}$$

$$= \sum \{\|z_\delta - x_j\| - \|z_\delta - x_{j+1}\| : \ j \in \{k\bar{N}, \dots, (k+1)\bar{N} - 1\}, \ \mu_{j+1} < \gamma_0\}$$

$$\geq \Delta\gamma - 2\delta\bar{q} - 2\delta\bar{q}(\bar{N} - 1) = \Delta\gamma - 2\delta\bar{q}\bar{N}. \tag{3.386}$$

It follows from (3.371), (3.385), and (3.386) that

$$2M \geq \sum \{ \sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (\|z_\delta - x_j\| - \|z_\delta - x_{j+1}\|) : \ k \in E_2 \cap [0, n-1]\}$$

$$+ \sum \{ \sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (\|z_\delta - x_j\| - \|z_\delta - x_{j+1}\|) : \ k \in \{0, \dots, n-1\} \setminus E_2\}$$

$$\geq \mathrm{Card}(E_2 \cap [0, n-1])(\Delta\gamma - 2\delta\bar{q}\bar{N}) - 2n\bar{N}\delta\bar{q}.$$

Since δ is any element of the interval $(0, 1)$ we conclude that

$$\mathrm{Card}(E_2 \cap [0, n-1]) \leq 2M(\Delta\gamma)^{-1}.$$

Since the relation above holds for every natural number n we conclude that

$$\mathrm{Card}(E_2) \leq 2M(\Delta\gamma)^{-1}. \tag{3.387}$$

In view of (3.384) and (3.387), there exists an integer $q_0 \geq 0$ such that

$$q_0 \leq 2M(\Delta\gamma)^{-1} + 1, \ q_0 \notin E_2 \tag{3.388}$$

and

$$\mu_{i+1} < \gamma_0, \ i = q_0\bar{N}, \dots, (q_0 + 1)\bar{N} - 1. \tag{3.389}$$

By (3.7), (3.17), (3.362), (3.363), (3.365), (3.366), and (3.389), for all integers $i = q_0\bar{N}, \dots, (q_0 + 1)\bar{N} - 1$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t) - 1$,

$$\gamma_0 > \mu_{i+1} \geq \beta_{i,t} \geq d(y_j^{(i,t)}, C_{t_{j+1}}), \tag{3.390}$$

there exists

$$\xi_{i,t,j} \in C_{t_{j+1}}$$

such that

$$\|y_j^{(i,t)} - \xi_{i,t,j}\| < \gamma_0$$

and

$$\begin{aligned} \|y_j^{(i,t)} - y_{j+1}^{(i,t)}\| &= \|y_j^{(i,t)} - P_{t_{j+1}}(y_j^{(i,t)})\| \\ &\le \|y_j^{(i,t)} - \xi_{i,t,j}\| + \|\xi_{i,t,j} - P_{t_{j+1}}(y_j^{(i,t)})\| \\ &\le 2\|y_j^{(i,t)} - \xi_{i,t,j}\| < 2\gamma_0 \end{aligned} \tag{3.391}$$

and

$$\begin{aligned} \|x_i - y_{i,t}\| &= \|y_0^{(i,t)} - y_{p(t)}^{(i,t)}\| \\ &\le \sum_{j=0}^{p(t)-1} \|y_j^{(i,t)} - y_{j+1}^{(i,t)}\| \le 2\bar{q}\gamma_0. \end{aligned} \tag{3.392}$$

It follows from (3.11), (3.360), (3.392), and the convexity of the function $\|\cdot\|$ that for all $i = q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1$,

$$\begin{aligned} &\|x_i - x_{i+1}\| \\ &\le \|x_i - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\| \\ &\le \sum_{t \in \Omega_{i+1}} w_{i+1}(t)\|x_i - y_{i,t}\| \le 2\bar{q}\gamma_0. \end{aligned} \tag{3.393}$$

In view of (3.393),

$$\|x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}}\| \le 2\bar{q}\gamma_0\bar{N}. \tag{3.394}$$

By (3.349), (3.354), (3.356), (3.357), and (3.394), for each integer $q > q_0$,

$$\begin{aligned} \|x_{q\bar{N}} - x_{(q+1)\bar{N}}\| &= \|T^{q-q_0}(x_{q_0\bar{N}}) - T^{q-q_0}(x_{(q_0+1)\bar{N}})\| \\ &\le \|x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}}\| \le 2\bar{q}\gamma_0\bar{N}. \end{aligned} \tag{3.395}$$

Let $q \geq q_0$ be an integer. In view of (3.394) and (3.395),

$$\|x_{q\bar{N}} - x_{(q+1)\bar{N}}\| \leq 2\bar{q}\gamma_0\bar{N}. \tag{3.396}$$

Let $\delta \in (0, 1)$. By (3.371) and (3.396),

$$\begin{aligned}
2\bar{q}\gamma_0\bar{N} &\geq \|x_{q\bar{N}} - x_{(q+1)\bar{N}}\| \\
&\geq \|z_\delta - x_{q\bar{N}}\| - \|z_\delta - x_{(q+1)\bar{N}}\| \\
&= \sum_{i=q\bar{N}}^{(q+1)\bar{N}-1} (\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\|) \\
&= \sum\{\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\| : \ i \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}, \ \mu_{i+1} \geq \epsilon_0\} \\
&+ \sum\{\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\| : \ i \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}, \ \mu_{i+1} < \epsilon_0\} \\
&\geq \sum\{\|z_\delta - x_i\| - \|z_\delta - x_{i+1}\| : \ i \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}, \ \mu_{i+1} \geq \epsilon_0\} - 2\delta\bar{q}\bar{N}.
\end{aligned} \tag{3.397}$$

Let

$$i \in \{0, 1, \dots, \}, \ \mu_{i+1} \geq \epsilon_0. \tag{3.398}$$

In view of (3.365), (3.366), and (3.398), there exists

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \Omega_{i+1}, \ j_0 \in \{1, \dots, p(\tau) - 1\}$$

such that

$$\epsilon_0 \leq \mu_{i+1} = \beta_{i,\tau} = d(y_{j_0}^{(i,\tau)}, C_{\tau_{j_0+1}}). \tag{3.399}$$

Property (P7), (3.344), (3.346), (3.363), (3.374), and (3.399) imply that

$$\begin{aligned}
\|y_{j_0+1}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| &= \|P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}) - z_{\delta,\tau_{j_0+1}}\| \\
&\leq \|y_{j_0}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| - \epsilon_1.
\end{aligned} \tag{3.400}$$

It follows from (3.367) and (3.400) that

$$\begin{aligned}
\|y_{j_0+1}^{(i,\tau)} - z_\delta\| &\leq \|y_{j_0+1}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| + \|z_{\delta,\tau_{j_0+1}} - z_\delta\| \\
&\leq \delta + \|y_{j_0}^{(i,\tau)} - z_{\delta,\tau_{j_0+1}}\| - \epsilon_1 \\
&\leq \|y_{j_0}^{(i,\tau)} - z_\delta\| + 2\delta - \epsilon_1.
\end{aligned} \tag{3.401}$$

By (3.17), (3.18), (3.362), (3.368), and (3.401),

$$\begin{aligned} &\|z_\delta - x_i\| - \|z_\delta - y_{i,\tau}\| \\ &= \|z_\delta - y_0^{(i,\tau)}\| - \|z_\delta - y_{p(\tau)}^{(i,\tau)}\| \\ &= \sum_{j=0}^{p(\tau)-1} [\|z_\delta - y_j^{(i,\tau)}\| - \|z_\delta - y_{j+1}^{(i,\tau)}\|] \\ &\geq \|z_\delta - y_{j_0}^{(i,\tau)}\| - \|z_\delta - y_{j_0+1}^{(i,\tau)}\| - 2\delta(p(\tau) - 1) \\ &\geq \epsilon_1 - 2\delta\bar{q}. \end{aligned} \tag{3.402}$$

By (3.11), (3.360), (3.370), (3.402), and the convexity of the norm,

$$\begin{aligned} &\|z_\delta - x_{i+1}\| \\ &= \|z_\delta - \sum_{t\in\Omega_{i+1}} w_{i+1}(t) y_{i,t}\| \\ &\leq \sum_{t\in\Omega_{i+1}} w_{i+1}(t)\|z_\delta - y_{i,t}\| \\ &= w_{i+1}(\tau)\|z_\delta - y_{i,\tau}\| \\ &+ \sum\{w_{i+1}(t)\|z_\delta - y_{i,t}\| :\ t \in \Omega_{i+1} \setminus \{\tau\}\} \\ &\leq w_{i+1}(\tau)(\|z_\delta - x_i\| - \epsilon_1 + 2\delta\bar{q}) \\ &+ \sum\{w_{i+1}(t)(\|z_\delta - x_i\| + 2\delta\bar{q}) :\ t \in \Omega_{i+1} \setminus \{\tau\}\} \\ &\leq \|z_\delta - x_i\| + 2\delta\bar{q} - \Delta\epsilon_1. \end{aligned}$$

Thus

$$\|z_\delta - x_{i+1}\| \leq \|z_\delta - x_i\| + 2\delta\bar{q} - \Delta\epsilon_1 \text{ for all integers } i \geq 0 \text{ such that } \mu_{i+1} \geq \epsilon_0. \tag{3.403}$$

Assume that there exists

$$i_0 \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$$

such that $\mu_{i_0+1} \geq \epsilon_0$. In view of (3.397) and (3.403),

$$2\bar{N}\bar{q}\gamma_0 \geq \Delta\epsilon_1 - 2\delta\bar{q} - 4\delta\bar{q}\bar{N}.$$

Since δ is any element of the interval (0, 1) the relation above implies that

$$\gamma_0 \geq \Delta\epsilon_1(2\bar{N}\bar{q})^{-1}.$$

This contradicts (3.400). The contradiction we have reached proves

$$\mu_{i+1} < \epsilon_0 \text{ for all } i \in \{q\bar{N}, \ldots, (q+1)\bar{N} - 1\}. \tag{3.404}$$

It follows from (3.7), (3.17), (3.362), (3.363), (3.365), (3.366), and (3.404) that for every $i \in \{q\bar{N}, \ldots, (q+1)\bar{N} - 1\}$, every $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and every $j = 0, \ldots, p(t) - 1$,

$$\epsilon_0 > \mu_{i+1} \geq \beta_{i,t} \geq d(y_j^{(i,t)}, C_{t_{j+1}}) \tag{3.405}$$

and there exists

$$\xi_{i,t,j} \in C_{t_{j+1}}$$

such that

$$\|y_j^{(i,t)} - \xi_{i,t,j}\| < \epsilon_0,$$

$$\|y_{j+1}^{(i,t)} - y_j^{(i,t)}\| \leq \|P_{t_{j+1}}(y_j^{(i,t)}) - \xi_{i,t,j}\| + \|\xi_{i,t,j} - y_j^{(i,t)}\|$$
$$\leq 2\|y_j^{(i,t)} - \xi_{i,t,j}\| < 2\epsilon_0, \tag{3.406}$$

$$\|x_i - y_j^{(i,t)}\| = \|y_0^{(i,t)} - y_j^{(i,t)}\| < 2j\epsilon_0 < 2\epsilon_0\bar{q}, \tag{3.407}$$

$$\|x_i - y_{i,t}\| = \|y_0^{(i,t)} - y_{p(t)}^{(i,t)}\|$$
$$\leq \sum_{j=0}^{p(t)-1} \|y_{j+1}^{(i,t)} - y_j^{(i,t)}\| \leq 2\epsilon_0\bar{q}. \tag{3.408}$$

It follows from (3.11), (3.360), (3.408), and the convexity of the norm that for all $i = q\bar{N}, \ldots, (q+1)\bar{N} - 1$

$$\|x_{i+1} - x_i\|$$
$$\leq \|x_i - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_{i,t}\|$$
$$\leq \sum_{t \in \Omega_{i+1}} w_{i+1}(t)\|y_{i,t} - x_i\| \leq 2\epsilon_0\bar{q}. \tag{3.409}$$

By (3.409), for each pair of integers $i_1, i_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$\|x_{i_1} - x_{i_2}\| \le 2\bar{q}\bar{N}\epsilon_0. \tag{3.410}$$

Let

$$k \in \{q\bar{N}, \dots, (q+1)\bar{N}\},\ s \in \{1, \dots, m\}. \tag{3.411}$$

In view of (3.353), there exist

$$j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1} \tag{3.412}$$

such that

$$s \in \{t_1, \dots, t_{p(t)}\}. \tag{3.413}$$

In view of (3.413), there exists $l \in \{0, \dots, p(t) - 1\}$ such that

$$s = t_{l+1}. \tag{3.414}$$

It follows from (3.347), (3.405), (3.407), and (3.410)–(3.412) that

$$\begin{aligned}
d(y_l^{(j,t)}, C_s) &\le \epsilon_0,\\
d(x_j, C_s) \le \|x_j - y_l^{(j,t)}\| + d(y_l^{(j,t)}, C_s) &\le \epsilon_0(2\bar{q}+1),\\
d(x_k, C_s) &\le \|x_k - x_j\| + d(x_j, C_s)\\
\le 2\bar{N}\bar{q}\epsilon_0 + \epsilon_0(2\bar{q}+1) &\le (\bar{N}+1)\epsilon_0(2\bar{q}+1) = \epsilon,\\
d(x_k, C_s) &\le \epsilon,\ s = 1, \dots, m
\end{aligned} \tag{3.415}$$

for every $k \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$ and for all integers $q \ge q_0$. Thus (3.415) holds for all integers $k \ge q_0\bar{N}$. Theorem 3.8 is proved.

3.13 Proof of Theorem 3.9

We may assume that $\epsilon_0 < r_0/2$. Theorem 3.9 is deduced from Theorems 2.9 and 3.8. Let $Y = X$, $\rho(y, z) = \|y - z\|$, $y, z \in X$, $\mathfrak{A}$ be the set of all mappings S defined on the set of natural numbers such that

$$S(i) = P_{\Omega_i, w_i},\ i = 1, 2, \dots,$$

where

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies

$$(\Omega_{i+\bar{N}}, w_{i+\bar{N}}) = (\Omega_i, w_i) \text{ for all integers } i \geq 1$$

and

$$\{1, \dots, m\} \subset \cup_{i=1}^{\bar{N}} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}).$$

Set

$$F = \{x \in X : \ d(x, C_s) \leq \epsilon/4, \ s = 1, \dots, m\}.$$

Theorem 3.8 implies that property (P6) holds.

Let $Q > 0$ be as guaranteed by property (P6) and

$$\delta = 4^{-1}\epsilon(Q(2\bar{N} + 1))^{-1}(\bar{q} + 1)^{-1}.$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}),$$

$$(\Omega_{i+\bar{N}}, w_{i+\bar{N}}) = (\Omega_i, w_i) \text{ for all integers } i \geq 0$$

and that sequences $\{x_i\}_{i=0}^{\infty} \subset X$, $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy

$$x_0 \in B(0, M),$$

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta) \text{ for all integers } i \geq 1.$$

Arguing as in the proof of Theorem 3.6 and using Proposition 2.8 we can show that for all integers $i \geq 0$,

$$\|x_{i+1} - P_{\Omega_{i+1}, w_{i+1}}(x_i)\| \leq (\bar{q} + 1)\delta \leq 4^{-1}\epsilon_0(Q(2\bar{N} + 1))^{-1}.$$

Theorem 2.9, the choice of Q and the relation above imply that for all integers $i \geq Q$,

$$B(x_i, \epsilon/4) \cap F \neq \emptyset$$

for all integers $i \geq Q$. This completes the proof of Theorem 3.9.

Chapter 4
Dynamic String-Maximum Methods in Metric Spaces

In this chapter we study the convergence of dynamic string-maximum methods for solving common fixed point problems in a metric space. Our main goal is to obtain an approximate solution of the problem using perturbed algorithms. We show that the inexact iterative method generates an approximate solution if perturbations are summable.

4.1 Preliminaries

Let (X, d) be a metric space. For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{d(x, y) : \ y \in E\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ d(x, y) \leq r\}.$$

Suppose that m is a natural number, $P_i : X \to X$, $i = 1, \ldots, m$ are self-mappings of X and that for every $i \in \{1, \ldots, m\}$,

$$\text{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \neq \emptyset. \tag{4.1}$$

We suppose that

$$d(z, x) \geq d(z, P_i(x)) \tag{4.2}$$

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for every $i \in \{1, \dots, m\}$, every $x \in X$, and every $z \in \mathrm{Fix}(P_i)$. For every $\epsilon > 0$ and every $i \in \{1, \dots, m\}$ put

$$F_\epsilon(P_i) = \{x \in X : \ d(x, P_i(x)) \leq \epsilon\}, \tag{4.3}$$

$$\tilde{F}_\epsilon(P_i) = \{y \in X : \ B(y, \epsilon) \cap F_\epsilon(P_i) \neq \emptyset\}, \tag{4.4}$$

$$F_\epsilon = \cap_{i=1}^m F_\epsilon(P_i) \tag{4.5}$$

and

$$\tilde{F}_\epsilon = \cap_{i=1}^m \tilde{F}_\epsilon(P_i) \tag{4.6}$$

A point belonging to the set F is a solution of our common fixed point problem while a point which belongs to the set $\tilde{F}_\epsilon$ is its ϵ-approximate solution.

Fix $\theta \in X$. Suppose that $M_* > 1$ and that the following assumption holds:

(A1) for each $\delta > 0$ there exists $z_\delta \in B(\theta, M_*)$ such that

$$B(z_\delta, \delta) \cap \ \mathrm{Fix}(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m.$$

We apply a dynamic string method with variable strings in order to obtain a good approximative solution of the common fixed point problem. Next we describe the dynamic string method with variable strings.

By an index vector, we mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q, \ \ P[t] = P_{t_q} \cdots P_{t_1}. \tag{4.7}$$

Denote by $\mathcal{M}$ the collection of all finite sets Ω of index vectors. Fix an integer

$$\bar{q} \geq m \tag{4.8}$$

and denote by $\mathcal{M}_*$ the set of all $\Omega \in \mathcal{M}$ such that

$$p(t) \leq \bar{q} \ \text{ for all } t \in \Omega. \tag{4.9}$$

The dynamic string-maximum method with variable strings can now be described by the following algorithm.

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick

$$\Omega_{k+1} \in \mathcal{M}_*,$$

calculate

$$P[t](x_k), \ \ t \in \Omega_{k+1}$$

and choose

$$x_{k+1} \in \{P[t](x_k) : t \in \Omega_{k+1}\}$$

such that

$$d(x_k, x_{k+1}) \ge d(x_k, P[t](x_k)), \ t \in \Omega_{k+1}.$$

In this chapter we use the following definitions.

Let $\delta \ge 0$, $x \in X$ and let $t = (t_1, \ldots, t_{p(t)})$ be an index vector. Define

$$\begin{aligned} A_0(x, t, \delta) = \{(y, \lambda) \in X \times R^1 : \ &\text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that} \\ &y_0 = x \text{ and for all } i = 1, \ldots, p(t), \\ &d(y_i, P_{t_i}(y_{i-1})) \le \delta, \\ &y = y_{p(t)}, \\ &\lambda = \max\{d(y_i, y_{i-1}) : i = 1, \ldots, p(t)\}\}. \end{aligned} \tag{4.10}$$

Let $\delta \ge 0$, $x \in X$ and let $\Omega \in \mathcal{M}_*$. Define

$$\begin{aligned} A(x, \Omega, \delta) = \{(y, \lambda) \in X \times R^1 : \ &\text{there exist} \\ &(y_t, \lambda_t) \in A_0(x, t, \delta), \ t \in \Omega \text{ such that} \\ &(y, \lambda) \in \{(y_t, \lambda_t) : t \in \Omega\}, \\ &\lambda \ge \lambda_t, \ t \in \Omega\}. \end{aligned} \tag{4.11}$$

Denote by $\mathrm{Card}(A)$ the cardinality of a set A. Suppose that the sum over empty set is zero.

4.2 The First Problem

We suppose that $\bar{c} \in (0, 1)$ and that for every $i \in \{1, \ldots, m\}$ the inequality

$$d(z, x)^2 \ge d(z, P_i(x))^2 + \bar{c}d(x, P_i(x))^2 \tag{4.12}$$

holds for all $x \in X$ and all $z \in \mathrm{Fix}(P_i)$.

Let $\bar{N}$ be a natural number.

In this chapter we prove the following result which shows that the inexact dynamic string-maximum method generates approximate solutions if perturbations are summable.

Theorem 4.1 *Let*

$$M > M_*, \ \epsilon \in (0, 1)$$

and let a sequence $\{\epsilon_i\}_{i=1}^{\infty} \subset (0, \infty)$ *satisfy*

$$\Lambda := \sum_{i=1}^{\infty} \epsilon_i < \infty. \tag{4.13}$$

Let a natural number n_0 *be such that for each integer* $i \geq n_0$,

$$\epsilon_i < \epsilon(\bar{N} + 1)^{-1}(1 + \bar{q})^{-1}. \tag{4.14}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{4.15}$$

satisfies for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}), \tag{4.16}$$

$$x_0 \in B(\theta, M) \tag{4.17}$$

and that sequences $\{x_i\}_{i=1}^{\infty} \subset X$, $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *satisfies for each natural number* i,

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, \epsilon_i). \tag{4.18}$$

Then

$$Card(\{i \in \{0, 1, \dots\} : \ x_i \notin \tilde{F}_\epsilon\})$$
$$\leq n_0 + \bar{N}(1 + \bar{N})^2(1 + \bar{q})^2\bar{c}^{-1}\epsilon^{-2}((4M + \Lambda\bar{q})^2 + \Lambda\bar{q}(6(4M + \bar{q}\Lambda) + 6)).$$

4.3 Proof of Theorem 3.1

By (A1), for every $\delta > 0$ there exists

$$z_\delta \in B(\theta, M_*) \tag{4.19}$$

such that

$$B(z_\delta, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset, \ i = 1, \dots, m. \tag{4.20}$$

In view of (4.20), for each $\delta > 0$ and each $i \in \{1, \dots, m\}$ there exists

$$z_{\delta,i} \in B(z_\delta, \delta) \cap \mathrm{Fixp}(P_i). \tag{4.21}$$

Let $i \geq 0$ be an integer. By (4.18),

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, \Omega_{i+1}, \epsilon_{i+1}). \tag{4.22}$$

By (4.11) and (4.22), there exist

$$(y_{i,t}, \alpha_{i,t}) \in A_0(x_i, t, \epsilon_{i+1}),\ t \in \Omega_{i+1} \tag{4.23}$$

such that

$$(x_{i+1}, \lambda_{i+1}) \in \{(y_{i,t}, \alpha_{i,t}) :\ t \in \Omega_{i+1}\}, \tag{4.24}$$

$$\lambda_{i+1} \geq \alpha_{i,t},\ t \in \Omega_{i+1}. \tag{4.25}$$

It follows from (4.10) and (4.23) that for every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(i,t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(i,t)} = x_i,\ y_{p(t)}^{(i,t)} = y_{i,t}, \tag{4.26}$$

for every integer $j = 1, \dots, p(t)$,

$$d(y_j^{(i,t)}, P_{t_j}(y_{j-1}^{(i,t)})) \leq \epsilon_{i+1}, \tag{4.27}$$

$$\alpha_{i,t} = \max\{d(y_{j+1}^{(i,t)}, y_j^{(i,t)}) :\ j = 0, \dots, p(t) - 1\}. \tag{4.28}$$

Let $\delta > 0$. By (4.17) and (4.19),

$$d(z_\delta, x_0) \leq 2M. \tag{4.29}$$

Let $i \geq 0$ be an integer,

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1},\ j = 0, \dots, p(t) - 1. \tag{4.30}$$

Relations (4.2), (4.21), and (4.27) imply that

$$\begin{aligned} d(z_\delta, y_{j+1}^{(i,t)}) &\leq d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)})) + d(P_{t_{j+1}}(y_j^{(i,t)}), y_{j+1}^{(i,t)}) \\ &\leq d(z_\delta, z_{\delta,t_{j+1}}) + d(z_{\delta,t_{j+1}}, P_{t_{j+1}}(y_j^{(i,t)})) + \epsilon_{i+1} \\ &\leq \delta + \epsilon_{i+1} + d(z_{\delta,t_{j+1}}, y_j^{(i,t)}) \\ &\leq d(z_\delta, y_j^{(i,t)}) + 2\delta + \epsilon_{i+1} \end{aligned}$$

and

$$d(z_\delta, y_{j+1}^{(i,t)}) \le d(z_\delta, y_j^{(i,t)}) + 2\delta + \epsilon_{i+1}. \tag{4.31}$$

Clearly,

$$\begin{aligned}
&d(z_\delta, y_j^{(i,t)})^2 - d(z_\delta, y_{j+1}^{(i,t)})^2 \\
&\ge d(z_\delta, y_j^{(i,t)})^2 - d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)}))^2 \\
&+d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)}))^2 - d(z_\delta, y_{j+1}^{(i,t)})^2 \\
&\ge d(z_\delta, y_j^{(i,t)})^2 - d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)}))^2 \\
&-d(y_{j+1}^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))(d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)})) + d(z_\delta, y_{j+1}^{(i,t)})).
\end{aligned} \tag{4.32}$$

It follows from (4.12) and (4.21) that

$$\begin{aligned}
&d(z_\delta, y_j^{(i,t)})^2 - d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)}))^2 \\
&\ge d(z_{\delta,t_{j+1}}, y_j^{(i,t)})^2 - d(z_{\delta,t_{j+1}}, P_{t_{j+1}}(y_j^{(i,t)}))^2 \\
&+d(z_\delta, y_j^{(i,t)})^2 - d(z_{\delta,t_{j+1}}, y_j^{(i,t)})^2 \\
&+d(z_{\delta,t_{j+1}}, P_{t_{j+1}}(y_j^{(i,t)}))^2 - d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)}))^2 \\
&\ge \bar{c} d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))^2 \\
&-d(z_\delta, z_{\delta,t_{j+1}})(d(z_{\delta,t_{j+1}}, y_j^{(i,t)}) + d(z_\delta, y_j^{(i,t)})) \\
&-d(z_\delta, z_{\delta,t_{j+1}})(d(z_{\delta,t_{j+1}}, P_{t_{j+1}}(y_j^{(i,t)})) + d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)}))).
\end{aligned} \tag{4.33}$$

By (4.2), (4.21), and (4.33),

$$\begin{aligned}
&d(z_\delta, y_j^{(i,t)})^2 - d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)}))^2 \\
&\ge \bar{c} d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))^2 \\
&-\delta(d(z_{\delta,t_{j+1}}, y_j^{(i,t)}) + d(z_\delta, y_j^{(i,t)})) \\
&-\delta(d(z_{\delta,t_{j+1}}, P_{t_{j+1}}(y_j^{(i,t)})) + d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)}))) \\
&\ge \bar{c} d(y_j^{(i,t}, P_{t_{j+1}}(y_j^{(i,t)}))^2 - \delta(2d(z_\delta, y_j^{(i,t)}) + \delta) \\
&-\delta(2d(z_{\delta,t_{j+1}}, P_{t_{j+1}}(y_j^{(i,t)})) + \delta)
\end{aligned}$$

$$\geq \bar{c}d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))^2 - \delta(2d(z_\delta, y_j^{(i,t)}) + \delta)$$
$$-\delta(2d(z_\delta, y_j^{(i,t)}) + 3\delta)$$
$$\geq \bar{c}d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))^2 - 2\delta(2d(z_\delta, y_j^{(i,t)}) + 3\delta). \tag{4.34}$$

It follows from (4.27), (4.32), and (4.34) that

$$d(z_\delta, y_j^{(i,t)})^2 - d(z_\delta, y_{j+1}^{(i,t)})^2$$
$$\geq \bar{c}d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))^2 - 2\delta(2d(z_\delta, y_j^{(i,t)}) + 3\delta)$$
$$-\epsilon_{i+1}(d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)})) + d(z_\delta, y_{j+1}^{(i,t)})). \tag{4.35}$$

In view of (4.2) and (4.21),

$$d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)})) \leq d(z_\delta, z_{\delta,t_{j+1}}) + d(z_{\delta,t_{j+1}}, P_{t_{j+1}}(y_j^{(i,t)}))$$
$$\leq \delta + d(z_{\delta,t_{j+1}}, y_j^{(i,t)})$$
$$\leq \delta + d(z_{\delta,t_{j+1}}, z_\delta) + d(z_\delta, y_j^{(i,t)})$$
$$\leq 2\delta + d(z_\delta, y_j^{(i,t)}). \tag{4.36}$$

In view of (4.27), (4.35), and (4.36),

$$d(z_\delta, y_j^{(i,t)})^2 - d(z_\delta, y_{j+1}^{(i,t)})^2$$
$$\geq \bar{c}d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))^2 - 2\delta(2d(z_\delta, y_j^{(i,t)}) + 3\delta)$$
$$-\epsilon_{i+1}(d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)})) + d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)})) + \epsilon_{i+1})$$
$$\geq \bar{c}d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))^2 - 2\delta(2d(z_\delta, y_j^{(i,t)}) + 3\delta)$$
$$-\epsilon_{i+1}(2d(z_\delta, y_j^{(i,t)}) + 4\delta + \epsilon_{i+1}). \tag{4.37}$$

It follows from (4.9), (4.26), and (4.31) that for all $j = 0, \ldots, p(t)$,

$$d(z_\delta, y_j^{(i,t)}) \leq d(z_\delta, y_0^{(i,t)}) + j(2\delta + \epsilon_{i+1})$$
$$\leq d(z_\delta, x_i) + p(t)(2\delta + \epsilon_{i+1})$$
$$\leq d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1}). \tag{4.38}$$

By (4.26) and (4.38),

$$d(z_\delta, y_{i,t}) = d(z_\delta, y^{(i,t)}_{p(t)}) \le d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1}). \tag{4.39}$$

By (4.24) and (4.39),

$$d(z_\delta, x_{i+1}) \le d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1}). \tag{4.40}$$

Set

$$\epsilon_0 = 0. \tag{4.41}$$

By induction we show that for all integers $i \ge 0$,

$$d(z_\delta, x_i) \le 2M + 2\bar{q}\delta i + (\sum_{j=0}^{i} \epsilon_j)\bar{q}. \tag{4.42}$$

In view of (4.29) and (4.41), inequality (4.42) is true for $i = 0$.

Assume that $i \ge 0$ is an integer and that (4.42) holds. By (4.40) and (4.42),

$$\begin{aligned} d(z_\delta, x_{i+1}) &\le d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1}) \\ &\le 2M + 2\bar{q}\delta(i+1) + (\sum_{j=0}^{i+1} \epsilon_j)\bar{q}. \end{aligned}$$

Therefore by induction we showed that (4.42) holds for all integers $i \ge 0$.

It follows from (4.38) and (4.42) that for all integers $i \ge 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t)$,

$$\begin{aligned} d(z_\delta, y^{(i,t)}_j) &\le d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1}) \\ &\le 2M + 2\bar{q}\delta(i+1) + (\sum_{j=0}^{i+1} \epsilon_j)\bar{q}. \end{aligned} \tag{4.43}$$

By (4.13), (4.19), (4.42), and (4.43), for all integers $i \ge 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t)$,

$$d(\theta, x_i) \le d(\theta, z_\delta) + d(x_i, z_\delta) \le 3M + 2\bar{q}\delta i + \Lambda\bar{q},$$

$$d(\theta, y^{(i,t)}_j) \le d(\theta, z_\delta) + d(y^{(i,t)}_j, z_\delta) \le 3M + 2\bar{q}\delta(i+1) + \Lambda\bar{q}.$$

Since the relation above holds for any $\delta > 0$ we conclude that for all integers $i \geq 0$, all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \ldots, p(t)$,

$$d(\theta, x_i) \leq 3M + \Lambda\bar{q}, \tag{4.44}$$

$$d(\theta, y_j^{(i,t)}) \leq 3M + \Lambda\bar{q}. \tag{4.45}$$

By (4.19), (4.37), and (4.45), for every $\delta > 0$, every integer $i \geq 0$, every $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$, and every $j = 0, \ldots, p(t) - 1$,

$$\begin{aligned} &d(z_\delta, y_j^{(i,t)})^2 - d(z_\delta, y_{j+1}^{(i,t)})^2 \\ &\geq \bar{c} d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))^2 \\ &-2\delta(2(4M + \bar{q}\Lambda) + 3\delta) - \epsilon_{i+1}(2(4M + \bar{q}\Lambda) + 4\delta + \epsilon_{i+1}). \end{aligned} \tag{4.46}$$

In view of (4.27) and (4.45), for every $\delta > 0$, every integer $i \geq 0$, every $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$, and every $j = 0, \ldots, p(t) - 1$,

$$\begin{aligned} &d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))^2 \geq d(y_j^{(i,t)}, y_{j+1}^{(i,t)})^2 \\ &-(d(y_j^{(i,t)}, y_{j+1}^{(i,t)})^2 - d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))^2) \\ &\geq d(y_j^{(i,t)}, y_{j+1}^{(i,t)})^2 \\ &-d(y_{j+1}^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))(d(y_j^{(i,t)}, y_{j+1}^{(i,t)}) + d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))) \\ &\geq d(y_j^{(i,t)}, y_{j+1}^{(i,t)})^2 \\ &-\epsilon_{i+1}(2d(y_j^{(i,t)}, y_{j+1}^{(i,t)}) + d(y_{j+1}^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))) \\ &\geq d(y_j^{(i,t)}, y_{j+1}^{(i,t)})^2 - \epsilon_{i+1}(4(3M + \bar{q}\Lambda) + \epsilon_{i+1}). \end{aligned} \tag{4.47}$$

It follows from (4.46) and (4.47) that for every $\delta > 0$, every integer $i \geq 0$, every $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and every $j = 0, \ldots, p(t) - 1$,

$$\begin{aligned} &d(z_\delta, y_j^{(i,t)})^2 - d(z_\delta, y_{j+1}^{(i,t)})^2 \\ &\geq \bar{c}\, d(y_j^{(i,t)}, y_{j+1}^{(i,t)})^2 - 2\delta(2(4M + \bar{q}\Lambda) + 3\delta) \\ &-\epsilon_{i+1}(6(4M + \bar{q}\Lambda) + 4\delta + 2\epsilon_{i+1}). \end{aligned} \tag{4.48}$$

Let $i \geq 0$ be an integer and $\delta \in (0, 1)$. By (4.9), (4.26), (4.28), and (4.48), for all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$,

$$\begin{aligned}
&d(z_\delta, x_i)^2 - d(z_\delta, y_{i,t})^2 \\
&= d(z_\delta, y_0^{(i,t)})^2 - d(z_\delta, y_{p(t)}^{(i,t)})^2 \\
&= \sum_{j=0}^{p(t)-1} (d(z_\delta, y_j^{(i,t)})^2 - d(z_\delta, y_{j+1}^{(i,t)})^2) \\
&\geq \bar{c} \sum_{j=0}^{p(t)-1} d(y_j^{(i,t)}, y_{j+1}^{(i,t)})^2 \\
&-2\delta\bar{q}(2(4M + \bar{q}\Lambda) + 3\delta) \\
&-\epsilon_{i+1}\bar{q}(6(4M + \bar{q}\Lambda) + 4\delta + 2\epsilon_{i+1}) \\
&\geq \bar{c}\alpha_{i,t}^2 - 2\delta\bar{q}(2(4M + \bar{q}\Lambda) + 3\delta) \\
&-\epsilon_{i+1}\bar{q}(6(4M + \bar{q}\Lambda) + 4\delta + 2\epsilon_{i+1}).
\end{aligned} \tag{4.49}$$

It follows from (4.24), (4.25), and (4.49) that

$$\begin{aligned}
&d(z_\delta, x_{i+1})^2 \\
&\leq d(z_\delta, x_i)^2 - \bar{c}\lambda_{i+1}^2 \\
&+2\delta\bar{q}(8M + 2\bar{q}\Lambda + 3\delta) + \epsilon_{i+1}\bar{q}(6(4M + \bar{q}\Lambda) + 4\delta + 2\epsilon_{i+1}).
\end{aligned} \tag{4.50}$$

Set

$$\gamma_0 = \epsilon(\bar{N} + 1)^{-1}(1 + \bar{q})^{-1}. \tag{4.51}$$

In view of (4.14) and (4.51), for all integers $i > n_0$,

$$\epsilon_i < \gamma_0. \tag{4.52}$$

It follows from (4.19), (4.44), and (4.50)–(4.52) that for each integer $n > n_0$,

$$\begin{aligned}
&(4M + \bar{q}\Lambda)^2 \geq d(z_\delta, x_{n_0})^2 \\
&\geq d(z_\delta, x_{n_0})^2 - d(z_\delta, x_n)^2 \\
&= \sum_{i=n_0}^{n-1} (d(z_\delta, x_i)^2 - d(z_\delta, x_{i+1})^2)
\end{aligned}$$

$$\geq \sum_{i=n_0}^{n-1} \bar{c}\lambda_{i+1}^2 - \sum_{i=n_0}^{n-1} \epsilon_{i+1}\bar{q}(6(4M+\bar{q}\Lambda)+4\delta+2\epsilon_{i+1})$$
$$-2(n-n_0)\delta\bar{q}(8M+2\bar{q}\Lambda+3\delta).$$

Since δ is any element of the interval $(0, 1)$, it follows from (4.13), (4.51), and (4.52) that for each integer $n > n_0$,

$$(4M+\bar{q}\Lambda)^2 + \bar{q}\Lambda(24M+6\Lambda\bar{q}+6)$$
$$\geq \sum_{i=n_0}^{n-1} \bar{c}\lambda_{i+1}^2$$
$$\geq \bar{c}\gamma_0^2 \mathrm{Card}(\{k \in \{n_0, \ldots, n-1\} : \lambda_{k+1} \geq \gamma_0\}).$$

Since the relation above holds for every natural number $n > n_0$ we conclude that

$$\mathrm{Card}(\{k \in \{n_0, n_0+1, \ldots, \} : \lambda_{k+1} \geq \gamma_0\})$$
$$\leq \bar{c}^{-1}\gamma_0^{-2}[(4M+\bar{q}\Lambda)^2 + \Lambda\bar{q}(24M+6\bar{q}\Lambda+6)]. \tag{4.53}$$

Assume that a natural number i satisfies

$$i \geq n_0, \ \lambda_{i+1} < \gamma_0. \tag{4.54}$$

Let $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$. By (4.25), (4.27), (4.28), and (4.54), for all $j = 0, \ldots, p(t)-1$,

$$\gamma_0 \geq d(y_{j+1}^{(i,t)}, y_j^{(i,t)})$$
$$\geq d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)})) - d(y_{j+1}^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))$$
$$\geq d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)})) - \epsilon_{i+1}. \tag{4.55}$$

In view of (4.52), (4.54), and (4.55), for all $j = 0, \ldots, p(t)-1$,

$$d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)})) < 2\gamma_0 \tag{4.56}$$

and

$$y_j^{(i,t)} \in F_{2\gamma_0}(P_{t_{j+1}}). \tag{4.57}$$

Relations (4.9), (4.26), (4.55), and (4.56) imply that for all $j = 0, \ldots, p(t)$,

$$d(x_i, y_j^{(i,t)}) = d(y_0^{(i,t)}, y_j^{(i,t)}) \leq j\gamma_0 \leq \bar{q}\gamma_0 \tag{4.58}$$

and if $j < p(t)$, then

$$x_i \in \tilde{F}_{(\bar{q}+1)\gamma_0}(P_{t_{j+1}}).$$

Therefore

$$x_i \in \tilde{F}_{(\bar{q}+1)\gamma_0}(P_{t_s}),\ s = 1, \ldots, p(t). \tag{4.59}$$

By (4.26) and (4.58),

$$d(x_i, y_{i,t}) \le \bar{q}\gamma_0 \text{ for all } t \in \Omega_{i+1}. \tag{4.60}$$

In view of (4.24) and (4.60),

$$d(x_i, x_{i+1}) \le \gamma_0\bar{q}. \tag{4.61}$$

Inclusion (4.59) implies that

$$x_i \in \cap\{\tilde{F}_{(\bar{q}+1)\gamma_0}(P_s) :\ s \in \cup_{t\in\Omega_{i+1}}\{t_1, \ldots, t_{p(t)}\}\}. \tag{4.62}$$

Let

$$E_0 = \{i \in \{n_0, n_0+1, \ldots\} :\ \lambda_{i+1} \ge \gamma_0\}. \tag{4.63}$$

It follows from (4.53) and (4.61)–(4.63) that

$$\mathrm{Card}(E_0) \le \bar{c}^{-1}\gamma_0^{-2}[(4M + \bar{q}\Lambda)^2 + \Lambda\bar{q}(24M + 6\bar{q}\Lambda + 6) \tag{4.64}$$

and the following property holds:

(P1) if a natural number $i \ge n_0$ satisfies $\lambda_{i+1} < \gamma_0$, then

$$d(x_{i+1}, x_i) \le \gamma_0\bar{q}, \tag{4.65}$$

$$x_i \in \cap\{\tilde{F}_{(\bar{q}+1)\gamma_0}(P_s),\ s \in \cup_{t\in\Omega_{i+1}}\{t_1, \ldots, t_{p(t)}\}\}. \tag{4.66}$$

Set

$$E_1 = \{i \in \{n_0, n_0+1, \ldots\} :\ [i, i+\bar{N}-1] \cap E_0 \ne \emptyset\}. \tag{4.67}$$

By (4.64) and (4.67),

$$\begin{aligned} \mathrm{Card}(E_1) &\le \bar{N}\mathrm{Card}(E_0) \\ &\le \bar{N}\bar{c}^{-1}\gamma_0^{-2}[(4M + \bar{q}\Lambda)^2 + \Lambda\bar{q}(24M + 6\bar{q}\Lambda + 6)]. \end{aligned} \tag{4.68}$$

Let an integer $j \ge n_0$ satisfy

$$j \notin E_1. \tag{4.69}$$

In view of (4.67) and (4.69),

$$[j, j + \bar{N} - 1] \cap E_0 = \emptyset. \tag{4.70}$$

Property (P1), (4.63), and (4.70) imply that for each $i \in \{j, \dots, j + \bar{N} - 1\}$, $\lambda_{i+1} < \gamma_0$ and (4.65) and (4.66) hold. It follows from (4.65) which holds for each $i \in \{j, \dots, j + \bar{N} - 1\}$ that for each pair of integers $i_1, i_2 \in \{j, \dots, j + \bar{N}\}$,

$$d(x_{i_1}, x_{i_2}) \le \bar{q}\bar{N}\gamma_0. \tag{4.71}$$

By (4.66) which holds for each $i \in \{j, \dots, j + \bar{N} - 1\}$, (4.16) and (4.71),

$$x_j \in \tilde{F}_{(\bar{q}+1)\gamma_0(\bar{N}+1)}(P_s),$$

$$s \in \cup_{i=j}^{j+\bar{N}-1} \cup \{\{t_1, \dots, t_{p(t)}\} : t = \{t_1, \dots, t_{p(t)}\} \in \Omega_{i+1}\} = \{1, \dots, m\}.$$

In view of the relation above and (4.51),

$$x_j \in \tilde{F}_{(\bar{q}+1)\gamma_0(\bar{N}+1)} = \tilde{F}_\epsilon$$

for all integers $j \ge n_0$ such that $j \notin E_1$. Together with (4.51) and (4.68) this implies that

$$\begin{aligned} &\mathrm{Card}(\{j \in \{0, 1, \dots\} : x_j \notin \tilde{F}_\epsilon\}) \\ &\le n_0 + \mathrm{Card}(E_1) \\ &\le n_0 + \bar{N}(1 + \bar{N})^2(1 + \bar{q})^2\bar{c}^{-1}\epsilon^{-2}((4M + \Lambda\bar{q})^2 + \Lambda\bar{q}(24M + 6\bar{q}\Lambda) + 6). \end{aligned}$$

Theorem 4.1 is proved.

4.4 The Second Problem

We suppose that the following assumption holds.

(A2) For each $M > 0$ and each $\gamma > 0$ there exists $\delta > 0$ such that for each $i \in \{1, \dots, m\}$, each

$$z \in B(\theta, M) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(\theta, M)$ satisfying $d(x, P_i(x)) \geq \gamma$, the inequality

$$d(P_i(x), z) \leq d(x, z) - \delta$$

is true.

In this chapter we prove the following result which shows that the inexact dynamic string-maximum method generates approximate solutions if perturbations are summable.

Theorem 4.2 *Let*

$$M \geq M_*, \ \epsilon \in (0, 1)$$

and let a sequence $\{\epsilon_i\}_{i=1}^{\infty} \subset (0, \infty)$ *satisfy*

$$\Lambda := \sum_{i=1}^{\infty} \epsilon_i < \infty. \tag{4.72}$$

Then there exists a natural number $Q > 0$ *such that for each*

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

which satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\})$$

and each pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$ *and* $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *which satisfies*

$$x_0 \in B(\theta, M)$$

and

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, \epsilon_i), \ i = 1, 2, \dots$$

the following inequality holds:

$$Card(\{i \in \{0, 1, \dots\} : \ x_i \notin \tilde{F}_\epsilon\}) \leq Q.$$

4.5 Proof of Theorem 4.2

By (A1), for every $\delta > 0$ there exists

$$z_\delta \in B(\theta, M_*) \tag{4.73}$$

such that

$$B(z_\delta, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset, \ i = 1, \dots, m. \tag{4.74}$$

In view of (4.74), for each $\delta > 0$ and each $i \in \{1, \dots, m\}$ there exists

$$z_{\delta,i} \in B(z_\delta, \delta) \cap \mathrm{Fixp}(P_i). \tag{4.75}$$

Set

$$\gamma_0 = \epsilon(\bar{N} + 1)^{-1}(\bar{q} + 1)^{-1}. \tag{4.76}$$

By (A2), there exists $\gamma \in (0, \gamma_0)$ such that the following property holds:

(P2) for each $i \in \{1, \dots, m\}$, each

$$z \in B(\theta, 3M + 1 + \Lambda) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(\theta, 3M + 1 + \Lambda(\bar{q} + 1))$ satisfying $d(x, P_i(x)) \geq \gamma_0/2$, the inequality

$$d(P_i(x), z) \leq d(x, z) - \gamma$$

is true.

In view of (4.72), there exists a natural number n_0 such that

$$\epsilon_i < \gamma/4 \text{ for all integers } i \geq n_0. \tag{4.77}$$

Set

$$Q = n_0 + 8\bar{N}\gamma^{-1}(M + (\bar{q} + 1)\Lambda). \tag{4.78}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{4.79}$$

satisfies for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}), \tag{4.80}$$

$\{x_i\}_{i=0}^{\infty} \subset X$ and $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy

$$x_0 \in B(\theta, M) \tag{4.81}$$

and

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, \epsilon_i),\ i = 1, 2, \dots, \tag{4.82}$$

Let $i \geq 0$ be an integer. By (4.82),

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, \Omega_{i+1}, \epsilon_{i+1}). \tag{4.83}$$

By (4.11) and (4.83), there exist

$$(y_{i,t}, \alpha_{i,t}) \in A_0(x_i, t, \epsilon_{i+1}),\ t \in \Omega_{i+1} \tag{4.84}$$

such that

$$(x_{i+1}, \lambda_{i+1}) \in \{(y_{i,t}, \alpha_{i,t}) :\ t \in \Omega_{i+1}\}, \tag{4.85}$$

$$\lambda_{i+1} \geq \alpha_{i,t},\ t \in \Omega_{i+1}. \tag{4.86}$$

It follows from (4.10) and (4.84) that for every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(i,t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(i,t)} = x_i,\ y_{p(t)}^{(i,t)} = y_{i,t}, \tag{4.87}$$

for every integer $j = 1, \dots, p(t)$,

$$d(y_j^{(i,t)}, P_{t_j}(y_{j-1}^{(i,t)})) \leq \epsilon_{i+1}, \tag{4.88}$$

$$\alpha_{i,t} = \max\{d(y_{j+1}^{(i,t)}, y_j^{(i,t)}) :\ j = 0, \dots, p(t) - 1\}. \tag{4.89}$$

Set

$$\epsilon_0 = 0. \tag{4.90}$$

Let $\delta > 0$. By (4.73) and (4.81),

$$d(z_\delta, x_0) \leq 2M. \tag{4.91}$$

Let $i \geq 0$ be an integer,

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1},\ j = 0, \dots, p(t) - 1. \tag{4.92}$$

Relations (4.2), (4.75), and (4.88) imply that

$$d(z_\delta, y_{j+1}^{(i,t)}) \leq d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)})) + d(P_{t_{j+1}}(y_j^{(i,t)}), y_{j+1}^{(i,t)})$$
$$\leq d(z_\delta, z_{\delta,t_{j+1}}) + d(z_{\delta,t_{j+1}}, P_{t_{j+1}}(y_j^{(i,t)})) + \epsilon_{i+1}$$

$$\leq \delta + \epsilon_{i+1} + d(z_{\delta,t_{j+1}}, y_j^{(i,t)})$$
$$\leq d(z_\delta, y_j^{(i,t)}) + 2\delta + \epsilon_{i+1}$$

and

$$d(z_\delta, y_{j+1}^{(i,t)}) \leq d(z_\delta, y_j^{(i,t)}) + 2\delta + \epsilon_{i+1}. \tag{4.93}$$

By (4.9), (4.87), and (4.93), for all $j = 0, \dots, p(t)$,

$$\begin{aligned} d(z_\delta, y_j^{(i,t)}) &\leq d(z_\delta, y_0^{(i,t)}) + j(2\delta + \epsilon_{i+1}) \\ &\leq d(z_\delta, x_i) + j(2\delta + \epsilon_{i+1}) \\ &\leq d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1}). \end{aligned} \tag{4.94}$$

By (4.87) and (4.94),

$$d(z_\delta, y_{i,t}) = d(z_\delta, y_{p(t)}^{(i,t)}) \leq d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1}). \tag{4.95}$$

In view of (4.85) and (4.95),

$$d(z_\delta, x_{i+1}) \leq d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1}). \tag{4.96}$$

By induction we show that for all integers $i \geq 0$,

$$d(z_\delta, x_i) \leq 2M + 2\bar{q}\delta i + (\sum_{j=0}^{i} \epsilon_j)\bar{q}. \tag{4.97}$$

In view of (4.90) and (4.91), inequality (4.97) is true for $i = 0$.

Assume that $i \geq 0$ is an integer and that (4.97) holds. By (4.96) and (4.97),

$$\begin{aligned} d(z_\delta, x_{i+1}) &\leq d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1}) \\ &\leq 2M + 2\bar{q}\delta(i+1) + (\sum_{j=0}^{i+1} \epsilon_j)\bar{q}. \end{aligned}$$

Therefore by induction we showed that (4.97) holds for all integers $i \geq 0$.

It follows from (4.94) and (4.97) that for all integers $i \geq 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t)$,

$$\begin{aligned} d(z_\delta, y_j^{(i,t)}) &\leq d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1}) \\ &\leq 2M + 2\bar{q}\delta(i+1) + (\sum_{j=0}^{i+1} \epsilon_j)\bar{q}. \end{aligned} \tag{4.98}$$

By (4.72), (4.73), (4.97), and (4.98), for all integers $i \geq 0$, all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \ldots, p(t)$,

$$d(\theta, x_i) \leq d(\theta, z_\delta) + d(x_i, z_\delta) \leq 3M + 2\bar{q}\delta i + \Lambda\bar{q},$$

$$d(\theta, y_j^{(i,t)}) \leq d(\theta, z_\delta) + d(y_j^{(i,t)}, z_\delta) \leq 3M + 2\bar{q}\delta(i+1) + \Lambda\bar{q}.$$

Since the relation above holds for any $\delta > 0$ we conclude that for all integers $i \geq 0$, all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \ldots, p(t)$,

$$d(\theta, x_i) \leq 3M + \Lambda\bar{q}, \tag{4.99}$$

$$d(\theta, y_j^{(i,t)}) \leq 3M + \Lambda\bar{q}. \tag{4.100}$$

Set

$$E_0 = \{i \in \{n_0, n_0 + 1, \ldots\} : \lambda_{i+1} \geq \gamma_0\}, \tag{4.101}$$

$$E_1 = \{n_0, n_0 + 1, \ldots, \} \setminus E_0.$$

Let

$$i \in E_0.$$

By (4.85) and (4.101),

$$\lambda_{i+1} \geq \gamma_0 \tag{4.102}$$

and there exists

$$\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \Omega_{i+1}$$

such that

$$x_{i+1} = y_{i,\tau},\ \alpha_{i,\tau} = \lambda_{i+1}. \tag{4.103}$$

By (4.89), (4.102), and (4.103), there exists

$$j_0 \in \{0, \ldots, p(\tau) - 1\} \tag{4.104}$$

such that

$$d(y_{j_0+1}^{(i,\tau)}, y_{j_0}^{(i,\tau)}) = \alpha_{i,\tau} = \lambda_{i+1} \geq \gamma_0. \tag{4.105}$$

In view of (4.88) and (4.105),

$$\begin{aligned} &d(y_{j_0}^{(i,\tau)}, P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)})) \\ &\geq d(y_{j_0+1}^{(i,\tau)}, y_{j_0}^{(i,\tau)}) - d(P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}), y_{j_0+1}^{(i,\tau)}) \\ &\geq \gamma_0 - \epsilon_{i+1}. \end{aligned} \tag{4.106}$$

It follows from (4.77), (4.101), and (4.106) that

$$d(y_{j_0}^{(i,\tau)}, P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)})) \geq \gamma_0 - \epsilon_{i+1} \geq \gamma_0 - \gamma/4 \geq \gamma_0/2. \tag{4.107}$$

Let $\delta \in (0, 1)$. Property (P2), (4.73), (4.75), (4.100), and (4.107) imply that

$$d(P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}), z_{\delta,\tau_{j_0+1}}) \leq d(y_{j_0}^{(i,\tau)}, z_{\delta,\tau_{j_0+1}}) - \gamma. \tag{4.108}$$

By (4.75), (4.77), (4.88), (4.101), and (4.108),

$$\begin{aligned} d(y_{j_0+1}^{(i,\tau)}, z_\delta) &\leq d(y_{j_0+1}^{(i,\tau)}, z_{\delta,\tau_{j_0+1}}) + d(z_{\delta,\tau_{j_0+1}}, z_\delta) \\ &\leq d(y_{j_0+1}^{(i,\tau)}, z_{\delta,\tau_{j_0+1}}) + \delta \\ &\leq d(y_{j_0+1}^{(i,\tau)}, P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)})) + d(P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}), z_{\delta,\tau_{j_0+1}}) + \delta \\ &\leq \epsilon_{i+1} + d(y_{j_0}^{(i,\tau)}, z_{\delta,\tau_{j_0+1}}) - \gamma + \delta \\ &\leq \gamma/4 - \gamma + \delta + d(y_{j_0}^{(i,\tau)}, z_\delta) + \delta, \\ d(y_{j_0+1}^{(i,\tau)}, z_\delta) &\leq 2\delta - 3\gamma/4 + d(y_{j_0}^{(i,\tau)}, z_\delta). \end{aligned} \tag{4.109}$$

By (4.9), (4.87), (4.93), (4.104), and (4.109),

$$\begin{aligned} &d(z_\delta, x_i) - d(z_\delta, y_{i,\tau}) \\ &= \sum_{j=0}^{p(\tau)-1} [d(z_\delta, y_j^{(i,\tau)}) - d(z_\delta, y_{j+1}^{(i,\tau)})] \\ &\geq d(y_{j_0}^{(i,\tau)}, z_\delta) - d(y_{j_0+1}^{(i,\tau)}, z_\delta) - (p(\tau) - 1)(2\delta + \epsilon_{i+1}) \\ &\geq 3\gamma/4 - 2\delta - (\bar{q} - 1)(2\delta + \epsilon_{i+1}).) \end{aligned} \tag{4.110}$$

It follows from (4.103) and (4.110) that

$$d(z_\delta, x_i) - d(z_\delta, x_{i+1}) \geq 3\gamma/4 - 2\delta - (\bar{q} - 1)(2\delta + \epsilon_{i+1}) \tag{4.111}$$

for each $i \in E_0$. By (4.72), (4.73), (4.96), (4.99), (4.101), and (4.111), for every integer $n > n_0$,

$$
\begin{aligned}
4M + \Lambda\bar{q} &\geq d(z_\delta, x_{n_0}) \\
&\geq d(z_\delta, x_{n_0}) - d(z_\delta, x_n) \\
&= \sum_{i=n_0}^{n-1} (d(z_\delta, x_i) - d(z_\delta, x_{i+1})) \\
&= \sum_{i \in E_0 \cap [0,n-1]} (d(z_\delta, x_i) - d(z_\delta, x_{i+1})) \\
&+ \sum_{i \in E_1 \cap [0,n-1]} (d(z_\delta, x_i) - d(z_\delta, x_{i+1}))
\end{aligned}
$$

$$
\begin{aligned}
&\geq \mathrm{Card}(E_0 \cap [0, n-1])(3\gamma/4) - 2\bar{q}\delta(n - n_0) - \bar{q}\sum_{i=0}^{n} \epsilon_i - \bar{q} \sum_{i \in E_1 \cap [0,n-1]} (2\delta + \epsilon_{i+1}) \\
&\geq \mathrm{Card}(E_0 \cap [0, n-1])(3\gamma/4) - 4\bar{q}\delta(n - n_0) - 2\bar{q}\Lambda.
\end{aligned}
$$

Since δ is any element of the interval $(0, 1)$ we conclude that

$$
(3\gamma/4)\mathrm{Card}(E_0 \cap [0, n-1]) \leq 4M + 3\Lambda\bar{q},
$$

$$
\mathrm{Card}(E_0 \cap [0, n-1]) \leq 2\gamma^{-1}(4M + 3\Lambda\bar{q}).
$$

Since n is any natural number satisfying $n > n_0$ we conclude that

$$
\mathrm{Card}(E_0) \leq 2\gamma^{-1}(4M + 3\Lambda\bar{q}). \tag{4.112}
$$

Assume that a natural number i satisfies

$$
i \geq n_0, \ \lambda_{i+1} < \gamma_0. \tag{4.113}
$$

Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$. By (4.77), (4.86), (4.88), (4.89), and (4.113), for all $j = 0, \dots, p(t) - 1$,

$$
\begin{aligned}
\gamma_0 > \lambda_{i+1} \geq \alpha_{i,t} &\geq d(y_{j+1}^{(i,t)}, y_j^{(i,t)}) \\
&\geq d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)})) - d(y_{j+1}^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)})) \\
&\geq d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)})) - \epsilon_{i+1},
\end{aligned} \tag{4.114}
$$

$$
d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)})) < \gamma_0 + \epsilon_{i+1} < 2\gamma_0 \tag{4.115}
$$

and

$$y_j^{(i,t)} \in F_{2\gamma_0}(P_{t_{j+1}}). \tag{4.116}$$

Relations (4.87) and (4.111) imply that for all $j = 0, \dots, p(t)$,

$$d(x_i, y_j^{(i,t)}) \le j\gamma_0 \le \bar{q}\gamma_0 \tag{4.117}$$

and if $j < p(t)$, then

$$x_i \in \tilde{F}_{(\bar{q}+1)\gamma_0}(P_{t_{j+1}}).$$

Therefore

$$x_i \in \tilde{F}_{(\bar{q}+1)\gamma_0}(P_{t_s}),\ s = 1, \dots, p(t),\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}. \tag{4.118}$$

By (4.87) and (4.117),

$$d(x_i, y_{i,t}) \le \bar{q}\gamma_0 \text{ for all } t \in \Omega_{i+1}. \tag{4.119}$$

It follows from (4.85) and (4.119) that

$$d(x_{i+1}, x_i) \le \bar{q}\gamma_0. \tag{4.120}$$

Thus we have shown that the following property holds:

(P3) if a natural number $i \ge n_0$ satisfies $\lambda_{i+1} < \gamma_0$, then (4.118) and (4.120) hold.

Set

$$E_2 = \{i \in \{n_0, n_0 + 1, \dots\} :\ [i, i + \bar{N} - 1] \cap E_0 \ne \emptyset\}. \tag{4.121}$$

By (4.113) and (4.121),

$$\begin{aligned} \mathrm{Card}(E_2) &\le \bar{N}\mathrm{Card}(E_0) \\ &\le 2\bar{N}\gamma^{-1}(4M + 3\bar{q}\Lambda). \end{aligned} \tag{4.122}$$

Let an integer $j \ge n_0$ satisfy

$$j \notin E_2. \tag{4.123}$$

Property (P3), (4.101), (4.121), and (4.123) imply that

$$[j, j + \bar{N} - 1] \cap E_0 = \emptyset,$$

for all $i = j, \dots, j + \bar{N} - 1$,

$$\lambda_{i+1} < \gamma_0$$

and that (4.118) and (4.120) hold. It follows from (4.120) which holds for each $i \in \{j, \dots, j + \bar{N} - 1\}$ that for each pair of integers $i_1, i_2 \in \{j, \dots, j + \bar{N}\}$,

$$d(x_{i_1}, x_{i_2}) \le \bar{q}\bar{N}\gamma_0. \tag{4.124}$$

By (4.80), (4.118) which holds for each $i \in \{j, \dots, j + \bar{N} - 1\}$ and (4.124),

$$x_j \in \tilde{F}_{(\bar{q}+1)\gamma_0(\bar{N}+1)}(P_s),$$

$$s \in \cup_{i=j}^{j+\bar{N}-1} \cup \{\{t_1, \dots, t_{p(t)}\} : t = \{t_1, \dots, t_{p(t)}\} \in \Omega_{i+1} = \{1, \dots, m\}.$$

In view of the relation above and (4.76),

$$x_j \in \tilde{F}_{(\bar{q}+1)\gamma_0(\bar{N}+1)} = \tilde{F}_\epsilon$$

for all integers $j \ge n_0$ such that $j \notin E_2$. Together with (4.78) and (4.122) this implies that

$$\begin{aligned} &\mathrm{Card}(\{j \in \{0, 1, \dots\} : \ x_j \notin \tilde{F}_\epsilon\}) \\ &\le n_0 + \mathrm{Card}(E_2) \\ &\le n_0 + 2\bar{N}(4M + 3\Lambda\bar{q})\gamma^{-1} \le Q. \end{aligned}$$

Theorem 4.2 is proved.

4.6 The Third Problem

For $i = 1, \dots, m$, set

$$C_i = \mathrm{Fix}(P_i). \tag{4.125}$$

We suppose that the following assumption holds.

(A3) For each $M > 0$ and each $\gamma > 0$ there exists $\delta > 0$ such that for each $i \in \{1, \dots, m\}$, each $x \in B(\theta, M)$ satisfying $d(x, C_i) \ge \gamma$ and each

$$z \in B(\theta, M) \cap C_i$$

the inequality

$$d(P_i(x), z) \le d(x, z) - \delta$$

is true.

In this chapter we prove the following result which shows that the inexact dynamic string-maximum method generates approximate solutions if perturbations are summable.

Theorem 4.3 *Let*

$$M > M_*, \ \epsilon \in (0, 1)$$

and let a sequence $\{\epsilon_i\}_{i=1}^{\infty} \subset (0, \infty)$ *satisfy*

$$\Lambda := \sum_{i=1}^{\infty} \epsilon_i < \infty. \tag{4.126}$$

Then there exists a number $Q > 0$ *such that for each*

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

which satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\})$$

and each pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$ *and* $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *which satisfies*

$$x_0 \in B(\theta, M)$$

and

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, \epsilon_i), \ i = 1, 2, \dots,$$

the following inequality holds:

$$Card(\{i \in \{0, 1, \dots\} : \ \max\{d(x_i, C_s) : \ s = 1, \dots, m\} > \epsilon\}) \le Q.$$

4.7 Proof of Theorem 4.3

By (A1), for every $\delta > 0$ there exists

$$z_\delta \in B(\theta, M_*) \tag{4.127}$$

such that

$$B(z_\delta, \delta) \cap C_i \neq \emptyset, \; i = 1, \dots, m. \tag{4.128}$$

In view of (4.128), for each $\delta > 0$ and each $i \in \{1, \dots, m\}$ there exists

$$z_{\delta,i} \in B(z_\delta, \delta) \cap C_i. \tag{4.129}$$

Set

$$\gamma_0 = \epsilon(\bar{N} + 1)^{-1}(3\bar{q} + 1)^{-1}. \tag{4.130}$$

By (A3), there exists $\gamma_1 \in (0, \gamma_0)$ such that the following property holds:

(P4) for each $i \in \{1, \dots, m\}$, each

$$z \in B(\theta, 3M + 1 + \Lambda) \cap C_i$$

and each $x \in B(\theta, 3M+1+\Lambda(\bar{q}+1))$ satisfying $d(x, C_i) \geq \gamma_0$, the inequality

$$d(P_i(x), z) \leq d(x, z) - \gamma_1/4$$

is true.

By (A3), there exists $\gamma \in (0, \gamma_1)$ such that the following property holds:

(P5) for each $i \in \{1, \dots, m\}$, each

$$z \in B(\theta, 3M + 1 + \Lambda) \cap C_i$$

and each $x \in B(\theta, 3M + 1 + \Lambda(\bar{q} + 1))$ satisfying $d(x, C_i) \geq \gamma_1/4$, the inequality

$$d(P_i(x), z) \leq d(x, z) - \gamma$$

is true.

In view of (4.126), there exists a natural number n_0 such that

$$\epsilon_i < \gamma/4 \text{ for all integers } i \geq n_0. \tag{4.131}$$

Set

$$Q = n_0 + 8\bar{N}\gamma^{-1}(M + (\bar{q} + 1)\Lambda). \tag{4.132}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{4.133}$$

satisfies for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}), \tag{4.134}$$

$\{x_i\}_{i=0}^{\infty} \subset X$ and $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy

$$x_0 \in B(\theta, M) \tag{4.135}$$

and

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, \epsilon_i), \ i = 1, 2, \dots, \tag{4.136}$$

Let $i \geq 0$ be an integer. By (4.136),

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, \Omega_{i+1}, \epsilon_{i+1}). \tag{4.137}$$

By (4.11) and (4.137), there exist

$$(y_{i,t}, \alpha_{i,t}) \in A_0(x_i, t, \epsilon_{i+1}), \ t \in \Omega_{i+1} \tag{4.138}$$

such that

$$(x_{i+1}, \lambda_{i+1}) \in \{(y_{i,t}, \alpha_{i,t}) : \ t \in \Omega_{i+1}\}, \tag{4.139}$$

$$\lambda_{i+1} \geq \alpha_{i,t}, \ t \in \Omega_{i+1}. \tag{4.140}$$

It follows from (4.10) and (4.138) that for every index vector

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$$

there exists a finite sequence $\{y_j^{(i,t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(i,t)} = x_i, \ y_{p(t)}^{(i,t)} = y_{i,t}, \tag{4.141}$$

for every integer $j = 1, \dots, p(t)$,

$$d(y_j^{(i,t)}, P_{t_j}(y_{j-1}^{(i,t)})) \leq \epsilon_{i+1}, \tag{4.142}$$

$$\alpha_{i,t} = \max\{d(y_{j+1}^{(i,t)}, y_j^{(i,t)}) : \ j = 0, \dots, p(t) - 1\}. \tag{4.143}$$

Set

$$\epsilon_0 = 0. \tag{4.144}$$

Let $\delta > 0$. By (4.127) and (4.135),

$$d(z_\delta, x_0) \le 2M. \tag{4.145}$$

Let $i \ge 0$ be an integer,

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1},\ \ j = 0, \dots, p(t) - 1. \tag{4.146}$$

Relations (4.2), (4.129), and (4.142) imply that

$$\begin{aligned} d(z_\delta, y_{j+1}^{(i,t)}) &\le d(z_\delta, P_{t_{j+1}}(y_j^{(i,t)})) + d(P_{t_{j+1}}(y_j^{(i,t)}), y_{j+1}^{(i,t)}) \\ &\le d(z_\delta, z_{\delta,t_{j+1}}) + d(z_{\delta,t_{j+1}}, P_{t_{j+1}}(y_j^{(i,t)})) + \epsilon_{i+1} \\ &\le \delta + \epsilon_{i+1} + \delta(z_{\delta,t_{j+1}}, y_j^{(i,t)}) \\ &\le d(z_\delta, y_j^{(i,t)}) + 2\delta + \epsilon_{i+1} \end{aligned}$$

and

$$d(z_\delta, y_{j+1}^{(i,t)}) \le d(z_\delta, y_j^{(i,t)}) + 2\delta + \epsilon_{i+1}. \tag{4.147}$$

By (4.9), (4.141), and (4.147), for all $j = 0, \dots, p(t)$,

$$\begin{aligned} d(z_\delta, y_j^{(i,t)}) &\le d(z_\delta, y_0^{(i,t)}) + j(2\delta + \epsilon_{i+1}) \\ &\le d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1}). \end{aligned} \tag{4.148}$$

By (4.141) and (4.148),

$$d(z_\delta, y_{i,t}) = d(z_\delta, y_{p(t)}^{(i,t)}) \le d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1}). \tag{4.149}$$

By (4.139) and (4.149),

$$d(z_\delta, x_{i+1}) \le d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1}). \tag{4.150}$$

By induction we show that for all integers $i \ge 0$,

$$d(z_\delta, x_i) \le 2M + 2\bar{q}\delta i + (\sum_{j=0}^{i} \epsilon_j)\bar{q}). \tag{4.151}$$

In view of (4.144) and (4.145), inequality (4.151) is true for $i = 0$.

Assume that $i \geq 0$ is an integer and that (4.151) holds. By (4.150) and (4.151),

$$d(z_\delta, x_{i+1}) \leq d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1})$$
$$\leq 2M + 2\bar{q}\delta(i+1) + (\sum_{j=0}^{i+1} \epsilon_j)\bar{q}.$$

Therefore by induction we showed that (4.151) holds for all integers $i \geq 0$.

It follows from (4.149) and (4.151) that for all integers $i \geq 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t)$,

$$d(z_\delta, y_j^{(i,t)}) \leq d(z_\delta, x_i) + \bar{q}(2\delta + \epsilon_{i+1})$$
$$\leq 2M + 2\bar{q}\delta(i+1) + (\sum_{j=0}^{i+1} \epsilon_j)\bar{q}. \tag{4.152}$$

By (4.126), (4.127), (4.151), and (4.152), for all integers $i \geq 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t)$,

$$d(\theta, x_i) \leq d(\theta, z_\delta) + d(x_i, z_\delta) \leq 3M + 2\bar{q}\delta i + \Lambda\bar{q},$$
$$d(\theta, y_j^{(i,t)}) \leq d(\theta, z_\delta) + d(y_j^{(i,t)}, z_\delta) \leq 3M + 2\bar{q}\delta(i+1) + \Lambda\bar{q}.$$

Since the relation above holds for any $\delta > 0$ we conclude that for all integers $i \geq 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $j = 0, \dots, p(t)$,

$$d(\theta, x_i) \leq 3M + \Lambda\bar{q}, \tag{4.153}$$

$$d(\theta, y_j^{(i,t)}) \leq 3M + \Lambda\bar{q}. \tag{4.154}$$

Set

$$E_0 = \{i \in \{n_0, n_0+1, \dots\} : \lambda_{i+1} \geq \gamma_1\}, \tag{4.155}$$

$$E_1 = \{n_0, n_0+1, \dots,\} \setminus E_0.$$

Fix $\delta \in (0, 1)$. We show that

$$d(z_\delta, x_i) - d(z_\delta, x_{i+1}) \geq (3/4)\gamma - \bar{q}\epsilon_{i+1} - 2\bar{q}\delta.$$

Let

$$i \in E_0.$$

By (4.139) and (4.155),

$$\lambda_{i+1} \geq \gamma_1 \tag{4.156}$$

and there exists

$$\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \Omega_{i+1}$$

such that

$$x_{i+1} = y_{i,\tau},\ \lambda_{i+1} = \alpha_{i,\tau}. \tag{4.157}$$

By (4.143), (4.156), and (4.157), there exists

$$j_0 \in \{1, \ldots, p(\tau) - 1\} \tag{4.158}$$

such that

$$d(y^{(i,\tau)}_{j_0+1}, y^{(i,\tau)}_{j_0}) = \alpha_{i,\tau} = \lambda_{i+1} \geq \gamma_1. \tag{4.159}$$

We show that

$$d(y^{(i,\tau)}_{j_0}, C_{\tau_{j_0+1}}) \geq \gamma_1/4.$$

There exists

$$\xi \in C_{\tau_{j_0+1}} \tag{4.160}$$

such that

$$d(y^{(i,\tau)}_{j_0}, \xi) \leq d(y^{(i,\tau)}_{j_0}, C_{\tau_{j_0+1}}) + \delta. \tag{4.161}$$

By (4.2), (4.142), (4.158), and (4.160),

$$\begin{aligned} d(y^{(i,\tau)}_{j_0+1}, \xi) &\leq d(y^{(i,\tau)}_{j_0+1}, P_{\tau_{j_0+1}}(y^{(i,\tau)}_{j_0})) + d(P_{\tau_{j_0+1}}(y^{(i,\tau)}_{j_0}), \xi) \\ &\leq \epsilon_{i+1} + d(y^{(i,\tau)}_{j_0}, \xi). \end{aligned} \tag{4.162}$$

In view of (4.161) and (4.162),

$$\begin{aligned} d(y^{(i,\tau)}_{j_0+1}, y^{(i,\tau)}_{j_0}) &\leq d(y^{(i,\tau)}_{j_0}, \xi) + d(\xi, y^{(i,\tau)}_{j_0+1}) \\ &\leq \epsilon_{i+1} + 2d(y^{(i,\tau)}_{j_0}, \xi) \\ &\leq \epsilon_{i+1} + 2d(y^{(i,\tau)}_{j_0}, C_{\tau_{j_0+1}}) + 2\delta. \end{aligned}$$

Since δ is any positive number from the interval (0, 1) it follows from (4.131), (4.155), and (4.159) that

$$\gamma/4 + 2d(y_{j_0}^{(i,\tau)}, C_{\tau_{j_0+1}}) \geq d(y_{j_0+1}^{(i,\tau)}, y_{j_0}^{(i,\tau)}) \geq \gamma_1,$$

$$d(y_{j_0}^{(i,\tau)}, C_{\tau_{j_0+1}}) \geq \gamma_1/4. \tag{4.163}$$

Property (P5), (4.127), (4.129), (4.154), (4.158), and (4.163) imply that

$$d(P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}), z_{\delta,\tau_{j_0+1}}) \leq d(y_{j_0}^{(i,\tau)}, z_{\delta,\tau_{j_0+1}}) - \gamma. \tag{4.164}$$

By (4.129), (4.142), and (4.164),

$$\begin{aligned} &d(y_{j_0+1}^{(i,\tau)}, z_\delta) \\ &\leq d(y_{j_0+1}^{(i,\tau)}, P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)})) + d(P_{\tau_{j_0+1}}(y_{j_0}^{(i,\tau)}), z_{\delta,\tau_{j_0+1}}) + d(z_{\delta,\tau_{j_0+1}}, z_\delta) \\ &\leq \epsilon_{i+1} + \delta + d(y_{j_0}^{(i,\tau)}, z_{\delta,\tau_{j_0+1}}) - \gamma \\ &\leq 2\delta + \epsilon_{i+1} + d(y_{j_0}^{(i,\tau)}, z_\delta) - \gamma. \end{aligned} \tag{4.165}$$

It follows from (4.9), (4.131), (4.141), (4.155), (4.157), (4.158), and (4.165) that

$$\begin{aligned} &d(z_\delta, x_i) - d(z_\delta, x_{i+1}) \\ &= d(z_\delta, x_i) - d(z_\delta, y_{i,\tau}) \\ &= \sum_{j=0}^{\tau-1}[d(z_\delta, y_j^{(i,\tau)}) - d(z_\delta, y_{j+1}^{(i,\tau)})] \\ &\geq d(y_{j_0}^{(i,\tau)}, z_\delta) - d(y_{j_0+1}^{(i,\tau)}, z_\delta) - (p(\tau) - 1)(2\delta + \epsilon_{i+1}) \\ &\geq \gamma - \epsilon_{i+1} - 2\delta - (p(\tau) - 1)(2\delta + \epsilon_{i+1}) \\ &\geq 3\gamma/4 - 2\delta\bar{q} - \bar{q}\epsilon_{i+1}. \end{aligned}$$

Thus we have shown that

$$d(z_\delta, x_i) - d(z_\delta, x_{i+1}) \geq 3\gamma/4 - 2\delta\bar{q} - \bar{q}\epsilon_{i+1} \tag{4.166}$$

for all $i \in E_0$.

By (4.50), (4.126), (4.127), (4.153), (4.155), and (4.166), for every integer $n > n_0$,

$$\begin{aligned} &(4M + \Lambda\bar{q}) \geq d(z_\delta, x_{n_0}) \\ &\geq d(z_\delta, x_{n_0}) - d(z_\delta, x_n) \end{aligned}$$

$$= \sum_{i=n_0}^{n-1} (d(z_\delta, x_i) - d(z_\delta, x_{i+1}))$$

$$= \sum_{i \in E_0 \cap [0,n-1]} (d(z_\delta, x_i) - d(z_\delta, x_{i+1}))$$

$$+ \sum_{i \in E_1 \cap [0,n-1]} (d(z_\delta, x_i) - d(z_\delta, x_{i+1}))$$

$$\geq \mathrm{Card}(E_0 \cap [0, n-1])(3\gamma/4) - 2\bar{q}\delta(n - n_0)$$

$$-\bar{q} \sum_{i=0}^{n} \epsilon_i - \bar{q} \sum_{i \in E_1 \cap [0,n-1]} (2\delta + \epsilon_{i+1})$$

$$\geq \mathrm{Card}(E_0 \cap [0, n-1])(3\gamma/4) - 4\bar{q}\delta(n - n_0) - 2\bar{q}\Lambda.$$

Since δ is any element of the interval (0, 1) we conclude that

$$(3\gamma/4)\mathrm{Card}(E_0 \cap [0, n-1]) \leq 4M + 3\Lambda\bar{q},$$

$$\mathrm{Card}(E_0 \cap [0, n-1]) \leq 2\gamma^{-1}(4M + 3\Lambda\bar{q}).$$

Since n is any natural number satisfying $n > n_0$ we conclude that

$$\mathrm{Card}(E_0) \leq 2\gamma^{-1}(4M + 3\Lambda\bar{q}). \tag{4.167}$$

Let $\delta \in (0, 1)$. Assume that an integer i satisfies

$$i \geq n_0, \ \lambda_{i+1} < \gamma_1.$$

Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$. By the relation above, (4.131), (4.140), (4.142), (4.143), and (4.168), for all $j = 0, \dots, p(t) - 1$,

$$\gamma_1 > \lambda_{i+1} \geq \alpha_{i,t} \geq d(y_j^{(i,t)}, y_{j+1}^{(i,t)})$$

$$\geq d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)})) - d(y_{j+1}^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)}))$$

$$\geq d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)})) - \epsilon_{i+1}, \tag{4.168}$$

$$d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)})) \leq \epsilon_{i+1} + \gamma_1 < 2\gamma_1. \tag{4.169}$$

Let $j \in \{0, \dots, p(t) - 1\}$. We show that

$$d(y_j^{(i,t)}, C_{t_{j+1}}) < \gamma_0.$$

Assume the contrary. Then

$$d(y_j^{(i,t)}, C_{t_{j+1}}) \geq \gamma_0. \tag{4.170}$$

In view of (4.127), (4.129), (4.154), and (4.170),

$$d(P_{t_{j+1}}(y_j^{(i,t)}), z_{\delta,t_{j+1}}) \leq d(y_j^{(i,t)}, z_{\delta,t_{j+1}}) - 4\gamma_1. \tag{4.171}$$

By (4.169) and (4.171),

$$\begin{aligned} 2\gamma_1 &> d(y_j^{(i,t)}, P_{t_{j+1}}(y_j^{(i,t)})) \\ &\geq d(y_j^{(i,t)}, z_{\delta,t_{j+1}}) - d(P_{t_{j+1}}(y_j^{(i,t)}), z_{\delta,t_{j+1}}) \geq 4\gamma_1, \end{aligned}$$

a contradiction. The contradiction we have reached proves that

$$d(y_j^{(i,t)}, C_{t_{j+1}}) < \gamma_0, \ \ j = 0, \ldots, p(t) - 1. \tag{4.172}$$

It follows from (4.9), (4.141), and (4.168) that for all $j = 0, \ldots, p(t)$,

$$d(y_j^{(i,t)}, x_i) \leq j\gamma_1 \leq \bar{q}\gamma_1. \tag{4.173}$$

Relations (4.172) and (4.173) imply that for all $j = 0, \ldots, p(t) - 1$,

$$d(x_i, C_{t_{j+1}}) < \bar{q}\gamma_1 + \gamma_0. \tag{4.174}$$

In view of (4.174),

$$\begin{aligned} &d(x_i, C_s) < \bar{q}\gamma_1 + \gamma_0 \\ &\text{for all } s \in \{t_1, \ldots, t_{p(t)}\} \text{ and all } (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}. \end{aligned} \tag{4.175}$$

By (4.139), (4.141), and (4.173),

$$d(x_i, x_{i+1}) \leq \bar{q}\gamma_1. \tag{4.176}$$

Thus we have shown that the following property holds:

(P6) for each integer $i \geq n_0$ satisfying $\lambda_{i+1} < \gamma_1$ inequalities (4.175) and (4.176) hold.

Set

$$E_2 = \{i \in \{n_0, n_0 + 1, \ldots\} : \ [i, i + \bar{N} - 1] \cap E_0 \neq \emptyset\}. \tag{4.177}$$

By (4.167) and (4.177),

$$\begin{aligned}\mathrm{Card}(E_2) &\le \bar{N}\mathrm{Card}(E_0)\\ &\le 2\bar{N}\gamma^{-1}(4M + 3\bar{q}\Lambda).\end{aligned} \tag{4.178}$$

Let an integer $j \ge n_0$ satisfy

$$j \notin E_2. \tag{4.179}$$

Property (P6), (4.155), (4.177), and (4.179) imply that

$$[j, j + \bar{N} - 1] \cap E_0 = \emptyset,$$

for each $i \in \{j, \dots, j + \bar{N} - 1\}$,

$$\lambda_{i+1} < \gamma_1$$

and that (4.175) and (4.176) hold. In view of (4.176), holding for all $i \in \{j, \dots, j + \bar{N} - 1\}$, we have that for all for each pair of integers $i_1, i_2 \in \{j, \dots, j + \bar{N} - 1\}$,

$$d(x_{i_1}, x_{i_2}) \le \bar{q}\bar{N}\gamma_1. \tag{4.180}$$

By (4.175), holding for all $i \in \{j, \dots, j + \bar{N} - 1\}$, and (4.180),

$$d(x_i, C_s) < \bar{q}\gamma_1(\bar{N} + 1) + \gamma_0 \tag{4.181}$$

for all $i \in \{j, \dots, j + \bar{N} - 1\}$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and all $s \in \{t_1, \dots, t_{p(t)}\}$. It follows from (4.130), (4.134), and (4.181) that

$$d(x_i, C_s) < (\bar{q} + 1)\gamma_0(\bar{N} + 1) \le \epsilon$$

for all $s \in \{1, \dots, m\}$ and all integers $j \ge n_0$ satisfying $j \notin E_2$. Together with (4.178) this implies that

$$\begin{aligned}&\mathrm{Card}(\{j \in \{0, 1, \dots\} : \max\{d(x_j, C_s) : s = 1, \dots, m\} \ge \epsilon\})\\ &\le n_0 + \mathrm{Card}(E_2)\\ &\le n_0 + 2\gamma^{-1}\bar{N}(4M + 3\Lambda\bar{q}) \le Q.\end{aligned}$$

Theorem 4.3 is proved.

Chapter 5
Abstract Version of CARP Algorithm

In this chapter we study the convergence of an abstract version of the algorithm which is called in the literature as component-averaged row projections or CARP. This algorithm was introduced for solving a convex feasibility problem in a finite-dimensional space, when a given collection of sets is divided into blocks in such a manner that all sets belonging to every block are subsets of a vector subspace associated with the block. All the blocks are processed in parallel and the algorithm operates in vector subspaces of the whole vector space. This method becomes efficient, in particular, when the dimensions of the subspaces are essentially smaller than the dimension of the whole space. In the chapter we study CARP for problems in a normed space, which is not necessarily finite-dimensional. Our main goal is to obtain an approximate solution of the problem using perturbed algorithms. We show that the inexact dynamic string-averaging algorithm generates an approximate solution if perturbations are summable. We also show that if the mappings are nonexpansive and the perturbations are sufficiently small, then the inexact dynamic string-averaging algorithm produces approximate solutions.

5.1 Preliminaries and Main Results

In [68] D. Gordon and R. Gordon studied a convex feasibility problem in a finite-dimensional space and introduced an algorithm which is called in the literature as component-averaged row projections or CARP. According to CARP, a given collection of sets is divided into blocks in such a manner that all sets belonging to every block are subsets of a vector subspace associated with the block. Here we study CARP for problems in a normed space, which is not necessarily finite-dimensional.

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Let $(Z, \|\cdot\|)$ be a normed space. For each $x \in Z$ and each $r > 0$ set

$$B_Z(x, r) = \{y \in Z : \|x - y\| \le r\}.$$

For each $x \in Z$ and each nonempty set $D \subset Z$ put

$$d_Z(x, D) = \inf\{\|x - y\| : y \in D\}.$$

Let $(Y_i, \|\cdot\|)$, $i = 1, \dots, p$ be normed spaces. The vector space

$$Y_1 \times \cdots \times Y_p = \prod_{i=1}^{p} Y_i$$

is equipped with the norm

$$\|y\| = \|(y_1, \dots, y_p)\| = (\sum_{i=1}^{p} \|y_i\|^2)^{1/2}, \; y = (y_1, \dots, y_p) \in \prod_{i=1}^{p} Y_i.$$

Suppose that $(X_i, \|\cdot\|)$, $i = 1, \dots, l$ are normed spaces and

$$X = \prod_{i=1}^{l} X_i.$$

Let m be a natural number,

$$C_i \subset X, \; i = 1, \dots, m$$

be nonempty closed subsets of X and let there exist a finite set $\mathcal{E}$ of index vectors $\tau = (\tau_1, \dots, \tau_p)$ such that

$$\tau_i \in \{1, \dots, m\} \text{ for each } i \in \{1, \dots, p\}, \tag{5.1}$$

$$\tau_{i_1} < \tau_{i_2} \text{ for each pair } i_1, i_2 \in \{\tau_1, \dots, \tau_p\} \text{ such that } i_1 < i_2, \tag{5.2}$$

$$\cup\{\{\tau_1, \dots, \tau_p\} : (\tau_1, \dots, \tau_p) \in \mathcal{E}\} = \{1, \dots, m\}. \tag{5.3}$$

For each $\tau = (\tau_1, \dots, \tau_q) \in \mathcal{E}$ set

$$p(\tau) = q. \tag{5.4}$$

Let $j \in \{1, \dots, l\}$. Denote by $\widehat{X}_j$ the set of all $x = (x_1, \dots, x_l) \in X$ such that $x_i = 0$ for all $i \in \{1, \dots, l\} \setminus \{j\}$. Clearly, $\widehat{X}_j$ is a vector subspace of X and it is equipped with the norm induced by the norm of X. Evidently, X_j and $\widehat{X}_j$ are isometric in a natural way.

Let $\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \mathcal{E}$. We suppose that there exists an index vector $\widehat{\tau} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_{p(\widehat{\tau})})$ such that

$$\widehat{\tau}_i \in \{1, \ldots, l\},\ i \in \{1, \ldots, p(\widehat{\tau})\}, \tag{5.5}$$

$$\widehat{\tau}_{i_1} < \widehat{\tau}_{i_2} \text{ for each pair } i_1, i_2 \in \{1, \ldots, p(\widehat{\tau})\} \text{ such that } i_1 < i_2 \tag{5.6}$$

and that for each $s \in \{\tau_1, \ldots, \tau_{p(\tau)}\}$ there exists a closed set

$$C_{\tau,s} \subset \sum\{\widehat{X}_i :\ i \in \{\widehat{\tau}_1, \ldots, \widehat{\tau}_{p(\widehat{\tau})}\}\} \tag{5.7}$$

such that

$$C_s = C_{\tau,s} + \sum\{\widehat{X}_i :\ i \in \{1, \ldots, l\} \setminus \{\widehat{\tau}_1, \ldots, \widehat{\tau}_{p(\widehat{\tau})}\}\}. \tag{5.8}$$

For each $\tau = (\tau_1, \ldots, \tau_p) \in \mathcal{E}$ set

$$\widehat{X}_\tau = \sum\{\widehat{X}_i :\ i \in \{\widehat{\tau}_1, \ldots, \widehat{\tau}_{p(\widehat{\tau})}\}\}. \tag{5.9}$$

We suppose that

$$\cup\,\{\{\widehat{\tau}_1, \ldots, \widehat{\tau}_{p(\widehat{\tau})}\} :\ \tau \in \mathcal{E}\} = \{1, \ldots, l\} \tag{5.10}$$

and that for each $\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \mathcal{E}$ and each $s \in \{\tau_1, \ldots, \tau_{p(\tau)})$ there exists a mapping

$$P_{\tau,s} : \widehat{X}_\tau \to \widehat{X}_\tau \tag{5.11}$$

such that

$$P_{\tau,s}(z) = z \text{ for all } z \in C_{\tau,s}, \tag{5.12}$$

$$\|z - x\| \geq \|z - P_{\tau,s}(x)\| \tag{5.13}$$

for all $z \in C_{\tau,s}$ and all $x \in \widehat{X}_\tau$.

We consider index vectors $t = (t_1, \ldots, t_p)$ such that $t_i \in \{1, \ldots, m\}$. For each index vector $t = (t_1, \ldots, t_q)$ set

$$p(t) = q. \tag{5.14}$$

Let $\bar{c} \in (0, 1)$. In this chapter we use the following assumptions.

(A1) For each $\tau = (\tau_1, \ldots, \tau_p) \in \mathcal{E}$ and each $s \in \{\tau_1, \ldots, \tau_p\}$,

$$P_{\tau,s}(\widehat{X}_\tau) = C_{\tau,s}, \tag{5.15}$$

$$\|z - x\|^2 \geq \|z - P_{\tau,s}(x)\|^2 + \bar{c}\|x - P_{\tau,s}(x)\|^2 \tag{5.16}$$

for all $z \in C_{\tau,s}$ and all $x \in \widehat{X}_\tau$.

(A2) For each $\Lambda > 0$ and each $\lambda > 0$ there exists $\gamma > 0$ such that for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$, each $z \in C_{\tau,s}$ satisfying $\|z\| \leq \Lambda$ and each $x \in \widehat{X}_\tau$ satisfying

$$\|x\| \leq \Lambda,\ d_{\widehat{X}_\tau}(x, C_{\tau,s}) \geq \lambda$$

the inequality

$$\|z - P_{\tau,s}(x)\| \leq \|z - x\| - \gamma$$

holds.

Let $\tau = (\tau_1, \dots, \tau_p) \in \mathcal{E}$. Consider a mapping $\pi_\tau : X \to \widehat{X}_\tau$ such that for each $x = (x_1, \dots, x_l) \in X$,

$$\pi_\tau(x) = (\pi_{\tau,1}(x), \dots, \pi_{\tau,l}(x)),$$

where for each $i \in \{1, \dots, l\}$,

$$\pi_{\tau,i}(x) = x_i \text{ if } i \in \{\widehat{\tau}_1, \dots, \widehat{\tau}_{p(\widehat{\tau})}\} \tag{5.17}$$

and

$$\pi_{\tau,i}(x) = 0 \text{ if } i \notin \{\widehat{\tau}_1, \dots, \widehat{\tau}_{p(\widehat{\tau})}\}. \tag{5.18}$$

We suppose that

$$C := \cap_{s=1}^m C_s \neq \emptyset.$$

Denote by Card(A) the cardinality of a set A. Suppose that the sum over empty set is zero.

For each $i \in \{1, \dots, l\}$ set

$$m_i = \text{Card}(\{\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E} :\ i \in \{\widehat{\tau}_1, \dots, \widehat{\tau}_{p(\widehat{\tau})}\}\}). \tag{5.19}$$

We consider linear operators

$$B_1 : X \to X,\ B_2 : X \to X$$

such that for each $i \in \{1, \dots, l\}$ and each $x \in \widehat{X}_i$,

$$B_1(x) = m_i^{-1}x,\ B_2(x) = m_i^{1/2}x. \tag{5.20}$$

Fix an integer

$$\bar{q} \geq m \text{ and } \Delta \in (0, m^{-1}]. \tag{5.21}$$

Let $\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \mathcal{E}$ and let an index vector $t = (t_1, \ldots, t_{p(t)})$ be such that

$$t_i \in \{\tau_1, \ldots, \tau_{p(\tau)}\},\ i = 1, \ldots, p(t).$$

Set

$$P[t] = P_{\tau, t_{p(t)}} \cdots P_{\tau, t_1}. \tag{5.22}$$

By (5.12), (5.13), and (5.22), for each $x \in \cap\{C_{\tau,s} :\ s \in \{\tau_1, \ldots, \tau_{p(\tau)}\}\}$,

$$P[t](x) = x \tag{5.23}$$

and for each $z \in \cap\{C_{\tau,s} :\ s \in \{\tau_1, \ldots, \tau_{p(\tau)}\}\}$, each $x \in \widehat{X}_\tau$,

$$\|P[t](x) - P[t](z)\| = \|P[t](x) - z\| \leq \|x - z\|. \tag{5.24}$$

Denote by $\mathcal{M}_\tau$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors $t = (t_1, \ldots, t_{p(t)})$ such that

$$t_i \in \{\tau_1, \ldots, \tau_{p(\tau)}\},\ i = 1, \ldots, p(t), \tag{5.25}$$

$$\cup\{\{t_1, \ldots, t_{p(t)}\} :\ t = (t_1, \ldots, t_{p(t)}) \in \Omega\} = \{\tau_1, \ldots, \tau_{p(\tau)}\}, \tag{5.26}$$

$$p(t) \leq \bar{q},\ t \in \Omega, \tag{5.27}$$

$$w : \Omega \to (0, \infty) \text{ satisfies } \sum_{t \in \Omega} w(t) = 1, \tag{5.28}$$

$$w(t) \geq \Delta,\ t \in \Omega. \tag{5.29}$$

Let $(\Omega, w) \in \mathcal{M}_\tau$. Define

$$P_{\Omega,w}(x) = \sum_{t \in \Omega} w(t) P[t](x),\ x \in \widehat{X}_\tau. \tag{5.30}$$

By (5.23), (5.24), (5.28), (5.30), and the convexity of the norm, for each $z \in \cap\{C_{\tau,s} :\ s \in \{\tau_1, \ldots, \tau_{p(\tau)}\}\}$ and each $x \in \widehat{X}_\tau$,

$$P_{\Omega,w}(z) = z, \tag{5.31}$$

$$\|P_{\Omega,w}(z) - P_{\Omega,w}(x)\| = \|z - P_{\Omega,w}(x)\| \leq \|z - x\|. \tag{5.32}$$

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm. Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick

$$(\Omega_{\tau,k+1}, w_{\tau,k+1}) \in \mathcal{M}_\tau,\ \ \tau \in \mathcal{E}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = B_1(\sum_{\tau \in \mathcal{E}}(P_{\Omega_{\tau,k+1}, w_{\tau,k+1}}(\pi_\tau(x_k))).$$

In order to state the main results of this chapter we need the following definitions.

Let $\delta \geq 0$, $x \in \widehat{X}_\tau$, $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, and $t = (t_1, \dots, t_{p(t)})$ be an index vector such that

$$t_i \in \{\tau_1, \dots, \tau_{p(\tau)}\},\ \ i = 1, \dots, p(t).$$

Define

$$\begin{aligned} A_{\tau,0}(x, t, \delta) = \{(y, \lambda) \in \widehat{X}_\tau \times R^1 :\\ \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset \widehat{X}_\tau \text{ such that}\\ y_0 = \pi_\tau(x) = x \text{ and for all } i = 1, \dots, p(t),\\ \|y_i - P_{\tau,t_i}(y_{i-1})\| \leq \delta,\\ y = y_{p(t)},\\ \lambda = \max\{\|y_i - y_{i-1}\| :\ i = 1, \dots, p(t)\}\}. \end{aligned} \tag{5.33}$$

Let $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, $x \in \widehat{X}_\tau$, $\delta \geq 0$ and let $(\Omega, w) \in \mathcal{M}_\tau$. Define

$$\begin{aligned} A_\tau(x, (\Omega, w), \delta) = \{(y, \lambda) \in \widehat{X}_\tau \times R^1 :\ \ \text{there exist}\\ (y_t, \lambda_t) \in A_{\tau,0}(x, t, \delta),\ \ t \in \Omega \text{ such that}\\ \|y - \sum_{t \in \Omega} w(t) y_t\| \leq \delta,\ \ \lambda = \max\{\lambda_t :\ t \in \Omega\}\}. \end{aligned} \tag{5.34}$$

Let $x \in X$, $\delta \geq 0$, $(\Omega_\tau, w_\tau) \in \mathcal{M}_\tau$, $\tau \in \mathcal{E}$. Define

$$\begin{aligned} A(x, \{(\Omega_\tau, w_\tau)\}_{\tau \in \mathcal{E}}, \delta) = \{(y, \lambda) \in X \times R^1 :\ \ \text{there exist}\\ (y_\tau, \lambda_\tau) \in A_\tau(\pi_\tau(x), (\Omega_\tau, w_\tau), \delta),\ \ \tau \in \mathcal{E} \text{ such that}\\ \|y - B_1(\sum_{\tau \in \mathcal{E}} y_\tau)\| \leq \delta,\ \ \lambda = \max\{\lambda_\tau :\ \tau \in \mathcal{E}\}\}. \end{aligned} \tag{5.35}$$

Set

$$m_0 = \max\{m_i : i = 1, \dots, l\}. \tag{5.36}$$

In this chapter we prove Theorems 5.1–5.6. In Theorems 5.1–5.3 we assume that (A1) holds. Theorem 5.1 shows that the inexact dynamic string-averaging method generates approximate solutions if perturbations are summable, Theorem 5.2 establishes that the exact dynamic string-averaging method generates approximate solutions, and Theorem 5.3 demonstrates that the inexact dynamic string-averaging method generates approximate solutions if the perturbations are small enough.

In Theorems 5.4–5.6 we assume that (A2) holds. Theorem 5.4 shows that the inexact dynamic string-averaging method generates approximate solutions if perturbations are summable, Theorem 5.5 establishes that the exact dynamic string-averaging method generates approximate solutions, and Theorem 5.6 demonstrates that the inexact dynamic string-averaging method generates approximate solutions if the perturbations are small enough.

Theorem 5.1 *Suppose that (A1) holds. Let $M > 0$ satisfy*

$$B_X(0, M) \cap C \neq \emptyset, \tag{5.37}$$

$\epsilon > 0$*, a sequence* $\{\epsilon_i\}_{i=1}^{\infty} \subset (0, \infty)$ *satisfy*

$$\Lambda := \sum_{i=1}^{\infty} \epsilon_i < \infty \tag{5.38}$$

and let

$$\tilde{M} = 8Mm_0^{1/2} + 4\Lambda((\bar{q} + 1)Card(\mathcal{E}) + m_0) + 2M. \tag{5.39}$$

Let a natural number n_0 be such that for each integer $i > n_0$,

$$\epsilon_i < \epsilon(2 + \bar{q})^{-1}. \tag{5.40}$$

Assume that for all natural numbers i,

$$(\Omega_{i,\tau}, w_{i,\tau}) \in \mathcal{M}_\tau, \ \tau \in \mathcal{E}, \tag{5.41}$$

$$x_0 \in B_X(0, M), \tag{5.42}$$

$\{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *and that for each natural number i,*

$$(x_i, \lambda_i) \in A(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau \in \mathcal{E}}, \epsilon_i). \tag{5.43}$$

Then

$$Card(\{i \in \{0,1,\dots\} : \max\{d_X(x_i, C_s) : s = 1,\dots,m\} > \epsilon\})$$
$$\le n_0 + \bar{c}^{-1}\Delta^{-1}\epsilon^{-2}(\bar{q}+2)^2[(2Mm_0^{1/2} + \Lambda((\bar{q}+1)Card(\mathcal{E}) + m_0)^2)$$
$$+\Lambda(2\bar{q}(\tilde{M}+1)Card(\mathcal{E}) + m_0^{1/2}(\tilde{M} + m_0^{1/2})].$$

Theorem 5.2 *Suppose that (A1) holds and that for each* $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$ *and each* $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$,

$$\|P_{\tau,s}(x_1) - P_{\tau,s}(x_2)\| \le \|x_1 - x_2\| \tag{5.44}$$

for all $x_1, x_2 \in \widehat{X}_\tau$. *Let* $M > 0$ *satisfy*

$$B_X(0, M) \cap C \ne \emptyset, \tag{5.45}$$

$\bar{N}$ *be a natural number and* $\epsilon \in (0,1)$. *Assume that for all natural numbers i,*

$$(\Omega_{i,\tau}, w_{i,\tau}) \in \mathcal{M}_\tau,\ \tau \in \mathcal{E}, \tag{5.46}$$

$$(\Omega_{i,\tau}, w_{i,\tau}) = (\Omega_{i+\bar{N},\tau}, w_{i+\bar{N},\tau}),\ \tau \in \mathcal{E}, \tag{5.47}$$

$$x_0 \in B_X(0, M) \tag{5.48}$$

and that sequence $\{x_i\}_{i=1}^\infty \subset X$, $\{\lambda_i\}_{i=1}^\infty \subset [0,\infty)$ *satisfy for each integer* $i \ge 1$,

$$(x_i, \lambda_i) \in A(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau\in\mathcal{E}}, 0). \tag{5.49}$$

Then for each integer

$$i \ge \bar{N}(4M^2 m_0 \bar{c}^{-1}\Delta^{-3}\epsilon^{-4}(\bar{q}+1)^4(4Mm_0 Card(\mathcal{E})^{1/2}\bar{N}\bar{q})^2 + 1)$$

the following relation holds:

$$d_X(x_i, C_s) \le \epsilon,\ s = 1,\dots,m.$$

Theorem 5.3 *Suppose that (A1) holds and that for each* $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$ *and each* $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$,

$$\|P_{\tau,s}(x_1) - P_{\tau,s}(x_2)\| \le \|x_1 - x_2\| \tag{5.50}$$

for all $x_1, x_2 \in \widehat{X}_\tau$. *Let*

$$C \ne \emptyset, \tag{5.51}$$

$M_0 > 0$, $r_0 \in (0, 1)$ satisfy

$$\{x \in X : \ d_X(x, C_s) \le r_0, \ s = 1, \dots, m\} \subset B_X(0, M_0), \tag{5.52}$$

$\bar{N}$ be a natural number, $\epsilon_0 \in (0, 1)$,

$$Q = 4^5 \bar{N} M_0^2 m_0 \bar{c}^{-1} \epsilon_0^{-2} (\bar{q} + 1)^4 \Delta^{-3} (4 M_0 m_0 Card(\mathcal{E})^{1/2} \bar{N} \bar{q})^2 + 1, \tag{5.53}$$

$$\delta = 4^{-1} \epsilon_0 Q^{-1} (2\bar{N} + 1)^{-1} m_0^{-1/2} (\bar{q} + 1)^{-1} Card(\mathcal{E})^{-1}. \tag{5.54}$$

Assume that for all natural numbers i,

$$(\Omega_{i,\tau}, w_{i,\tau}) \in \mathcal{M}_\tau, \ \tau \in \mathcal{E}, \tag{5.55}$$

$$(\Omega_{i,\tau}, w_{i,\tau}) = (\Omega_{i+\bar{N},\tau}, w_{i+\bar{N},\tau}), \ \tau \in \mathcal{E}, \tag{5.56}$$

$$x_0 \in B_X(0, M) \tag{5.57}$$

and that sequences $\{x_i\}_{i=1}^\infty \subset X$, $\{\lambda_i\}_{i=1}^\infty \subset [0, \infty)$ satisfy for each integer $i \ge 1$,

$$(x_i, \lambda_i) \in A(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau \in \mathcal{E}}, \delta). \tag{5.58}$$

Then for each integer $i \ge Q$,

$$d_X(x_i, C_s) \le \epsilon, \ s = 1, \dots, m.$$

Theorem 5.4 *Suppose that (A2) holds. Let $M > 0$ satisfy*

$$B_X(0, M) \cap C \neq \emptyset, \tag{5.59}$$

$\epsilon \in (0, 1)$ and let a sequence $\{\epsilon_i\}_{i=1}^\infty \subset (0, \infty)$ satisfy

$$\Lambda := \sum_{i=1}^\infty \epsilon_i < \infty. \tag{5.60}$$

Then there exists a number $Q > 0$ such that for each

$$(\Omega_{i,\tau}, w_{i,\tau}) \in \mathcal{M}_\tau, \ \tau \in \mathcal{E}, \ i = 1, 2, \dots,$$

each

$$x_0 \in B_X(0, M),$$

each $\{x_i\}_{i=1}^\infty \subset X$ and each $\{\lambda_i\}_{i=1}^\infty \subset [0, \infty)$ satisfying for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau \in \mathcal{E}}, \epsilon_i)$$

the inequality

$$Card(\{i \in \{0, 1, \dots\} : \max\{d_X(x_i, C_s) : s = 1, \dots, m\} > \epsilon\}) \le Q$$

holds.

Theorem 5.5 *Suppose that (A2) holds and that for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$ and each $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$,*

$$\|P_{\tau,s}(x_1) - P_{\tau,s}(x_2)\| \le \|x_1 - x_2\| \tag{5.61}$$

for all $x_1, x_2 \in \widehat{X}_\tau$. Let $M > 0$ satisfy

$$B_X(0, M) \cap C \ne \emptyset, \tag{5.62}$$

$\bar{N}$ be a natural number and $\epsilon \in (0, 1)$. Then there exists a number $Q > 0$ such that for each

$$(\Omega_{i,\tau}, w_{i,\tau}) \in \mathcal{M}_\tau,\ \tau \in \mathcal{E},\ i = 1, 2, \dots$$

satisfying for all integers $i \ge 1$,

$$(\Omega_{i,\tau}, w_{i,\tau}) = (\Omega_{i+\bar{N},\tau}, w_{i+\bar{N},\tau}),\ \tau \in \mathcal{E},$$

each

$$x_0 \in B_X(0, M),$$

each sequence $\{x_i\}_{i=1}^\infty \subset X$ and each sequence $\{\lambda_i\}_{i=1}^\infty \subset [0, \infty)$ satisfying for each integer $i \ge 1$,

$$(x_i, \lambda_i) \in A(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau \in \mathcal{E}}, 0)$$

the inequalities

$$d_X(x_i, C_s) \le \epsilon,\ s = 1, \dots, m$$

hold for all integers $i \ge Q$.

Theorem 5.6 *Suppose that (A2) holds and that for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$ and each $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$,*

$$\|P_{\tau,s}(x_1) - P_{\tau,s}(x_2)\| \le \|x_1 - x_2\| \tag{5.63}$$

for all $x_1, x_2 \in \widehat{X}_\tau$. Let

$$C \ne \emptyset, \tag{5.64}$$

$M_0 > 0$, $r_0 \in (0, 1)$ satisfy

$$\{x \in X : d_X(x, C_s) \le r_0, \ s = 1, \dots, m\} \subset B_X(0, M_0), \tag{5.65}$$

$\bar{N}$ be a natural number, $\epsilon_0 \in (0, r_0)$. Then there exist $Q, \delta > 0$ such that for each

$$(\Omega_{i,\tau}, w_{i,\tau}) \in \mathcal{M}_\tau, \ \tau \in \mathcal{E}, \ i = 1, 2, \dots$$

satisfying for all integers $i \ge 1$,

$$(\Omega_{i,\tau}, w_{i,\tau}) = (\Omega_{i+\bar{N},\tau}, w_{i+\bar{N},\tau}), \ \tau \in \mathcal{E},$$

each

$$x_0 \in B_X(0, M),$$

each sequence $\{x_i\}_{i=1}^\infty \subset X$ and each sequence $\{\lambda_i\}_{i=1}^\infty \subset [0, \infty)$ satisfying for each integer $i \ge 1$,

$$(x_i, \lambda_i) \in A(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau \in \mathcal{E}}, \delta)$$

the inequalities

$$d_X(x_i, C_s) \le \epsilon, \ s = 1, \dots, m$$

holds for all integers $i \ge Q$.

5.2 Auxiliary Results

Lemma 5.7 (Lemma 7.5 of [124]) *Let $z, x \in X$. Then*

$$\sum_{\tau \in \mathcal{E}} \|\pi_\tau(z) - \pi_\tau(x)\|^2 = \|B_2(z - x)\|^2.$$

Proof Let

$$z = (z_1, \dots, z_l\}, \ x = (x_1, \dots, x_l)$$

and

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}.$$

By the definition of π_τ (see (5.17), (5.18)),

$$\|\pi_\tau(z) - \pi_\tau(x)\|^2 = \sum \{\|z_i - x_i\|^2 : \ i \in \{\widehat{\tau}_1, \dots, \widehat{\tau}_{p(\widehat{\tau})}\}\}. \tag{5.66}$$

In view of (5.19), (5.20), and (5.66),

$$\sum_{\tau\in\mathcal{E}} \|\pi_\tau(z) - \pi_\tau(x)\|^2 = \sum_{i=1}^{l} \|z_i - x_i\|^2 m_i = \|B_2(z - x)\|^2.$$

Lemma 5.7 is proved. □

Lemma 5.8 (Lemma 7.6 of [124]) *Let* $z \in X$, $x_\tau \in \widehat{X}_\tau$, $\tau \in \mathcal{E}$,

$$x = B_1(\sum_{\tau\in\mathcal{E}} x_\tau). \tag{5.67}$$

Then

$$\|B_2(z - x)\|^2 \le \sum_{\tau\in\mathcal{E}} \|\pi_\tau(z) - x_\tau\|^2.$$

Proof Let

$$z = (z_1, \dots, z_l\},\ \ x_\tau = (x_{\tau,1}, \dots, x_{\tau,l}),\ \ \tau \in \mathcal{E}$$

and

$$x = (x_1, \dots, x_l).$$

In view of (5.20) and (5.67), for each $i \in \{1, \dots, l\}$,

$$x_i = m_i^{-1}(\sum\{x_{\tau,i} :\ \tau \in \mathcal{E},\ i \in \{\widehat{\tau}_1, \dots, \widehat{\tau}_{p(\widehat{\tau})}\}\}). \tag{5.68}$$

By (5.19), (5.68), and the convexity of the function $\|\cdot\|^2$, for each $i \in \{1, \dots, l\}$,

$$\begin{aligned}
\|z_i - x_i\|^2 &= \|z_i - m_i^{-1}(\sum\{x_{\tau,i} :\ \tau \in \mathcal{E},\ i \in \{\widehat{\tau}_1, \dots, \widehat{\tau}_{p(\widehat{\tau})}\}\})\|^2\\
&= \|\sum\{m_i^{-1}(z_i - x_{\tau,i}) :\ \tau \in \mathcal{E},\ i \in \{\widehat{\tau}_1, \dots, \widehat{\tau}_{p(\widehat{\tau})}\}\}\|^2\\
&\le \sum\{m_i^{-1}\|z_i - x_{\tau,i}\|^2 :\ \tau \in \mathcal{E},\ i \in \{\widehat{\tau}_1, \dots, \widehat{\tau}_{p(\widehat{\tau})}\}\},\\
m_i\|z_i - x_i\|^2 &\le \sum\{\|z_i - x_{\tau,i}\|^2 :\ \tau \in \mathcal{E},\ i \in \{\widehat{\tau}_1, \dots, \widehat{\tau}_{p(\widehat{\tau})}\}\}.
\end{aligned} \tag{5.69}$$

It follows from (5.20) and (5.69) that

$$\|B_2(z - x)\|^2 = \sum_{i=1}^{l} m_i\|z_i - x_i\|^2$$

$$\leq \sum_{i=1}^{l} \sum \{\|z_i - x_{\tau,i}\|^2 : \tau \in \mathcal{E}, \ i \in \{\widehat{\tau}_1, \dots, \widehat{\tau}_{p(\widehat{\tau})}\}\} = \sum_{\tau \in \mathcal{E}} \|\pi_\tau(z) - x_\tau\|^2.$$

Lemma 5.8 is proved.

5.3 Proof of Theorem 5.1

Set

$$\epsilon_0 = 0. \tag{5.70}$$

By (5.37), there exists

$$z_* = (z_{*,1}, \dots, z_{*,l}) \in B_X(0, M) \cap C. \tag{5.71}$$

Let $i \geq 0$ be an integer. By (5.43),

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, \{(\Omega_{i+1,\tau}, w_{i+1,\tau})\}_{\tau \in \mathcal{E}}, \epsilon_{i+1}). \tag{5.72}$$

In view of (5.35) and (5.72), there exist

$$(y_{i,\tau}, \lambda_{i,\tau}) \in A_\tau(\pi_\tau(x_i), (\Omega_{i+1,\tau}, w_{i+1,\tau}), \epsilon_{i+1}), \ \tau \in \mathcal{E} \tag{5.73}$$

such that

$$\|x_{i+1} - B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau})\| \leq \epsilon_{i+1}, \ \lambda_{i+1} = \max\{\lambda_{i,\tau} : \tau \in \mathcal{E}\}. \tag{5.74}$$

By (5.34) and (5.73), for each $\tau \in \mathcal{E}$, there exist

$$(y_t^{(i,\tau)}, \lambda_t^{(i,\tau)}) \in A_{\tau,0}(\pi_\tau(x_i), t, \epsilon_{i+1}), \ t \in \Omega_{i+1,\tau} \tag{5.75}$$

such that

$$\|y_{i,\tau} - \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\| \leq \epsilon_{i+1}, \tag{5.76}$$

$$\lambda_{i,\tau} = \max\{\lambda_t^{(i,\tau)} : t \in \Omega_{i+1,\tau}\}. \tag{5.77}$$

By (5.33) and (5.75), for each $\tau \in \mathcal{E}$ and each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ there exists a sequence

$$\{y_{t,j}^{(i,\tau)}\}_{j=0}^{p(t)} \subset \widehat{X}_\tau$$

such that

$$y_{t,0}^{(i,\tau)} = \pi_\tau(x_i), \tag{5.78}$$

for all $j = 1, \dots, p(t)$,

$$\|y_{t,j}^{(i,\tau)} - P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)})\| \le \epsilon_{i+1}, \tag{5.79}$$

$$y_t^{(i,\tau)} = y_{t,p(t)}^{(i,\tau)}, \tag{5.80}$$

$$\lambda_t^{(i,\tau)} = \max\{\|y_{t,j}^{(i,\tau)} - y_{t,j-1}^{(i,\tau)}\| : \ j = 1, \dots, p(t)\}. \tag{5.81}$$

In view of (5.20), (5.42), and (5.71),

$$\|z_* - x_0\| \le 2M,$$
$$\|B_2(z_* - x_0)\| \le 2M \max\{m_i^{1/2} : \ i = 1, \dots, l\}. \tag{5.82}$$

Let

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}, \ t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}.$$

By (5.7), (5.8), (5.13), (5.71), and (5.79), for each integer j satisfying $0 \le j < p(t)$,

$$\|\pi_\tau(z_*) - y_{t,j+1}^{(i,\tau)}\|$$
$$\le \|\pi_\tau(z_*) - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\| + \|P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)}) - y_{t,j+1}^{(i,\tau)}\|$$
$$\le \|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\| + \epsilon_{i+1}. \tag{5.83}$$

It follows from (5.27), (5.78), and (5.83) that for all integers $j \in \{0, \dots, p(t)\}$,

$$\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\| \le \|\pi_\tau(z_*) - y_{t,0}^{(i,\tau)}\| + j\epsilon_{i+1}$$
$$\le \|\pi_\tau(z_*) - \pi_\tau(x_i)\| + \bar{q}\epsilon_{i+1}. \tag{5.84}$$

Relations (5.80) and (5.84) imply that

$$\|\pi_\tau(z_*) - y_t^{(i,\tau)}\| \le \|\pi_\tau(z_*) - \pi_\tau(x_i)\| + \bar{q}\epsilon_{i+1}. \tag{5.85}$$

By (5.28), (5.76), (5.85), and the convexity of the norm,

$$\|\pi_\tau(z_*) - y_{i,\tau}\|$$
$$\le \|\pi_\tau(z_*) - \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\| + \|\sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)} - y_{i,\tau}\|$$

$$\leq \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) \|\pi_\tau(z_*) - y_t^{(i,\tau)}\| + \epsilon_{i+1}$$

$$\leq \|\pi_\tau(z_*) - \pi_\tau(x_i)\| + (\bar{q}+1)\epsilon_{i+1}. \tag{5.86}$$

Lemmas 5.7 and 5.8 and (5.86) imply that

$$\|B_2(z_* - B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau}))\|^2$$

$$\leq \sum_{\tau \in \mathcal{E}} \|\pi_\tau(z_*) - y_{i,\tau}\|^2$$

$$\leq \sum_{\tau \in \mathcal{E}} (\|\pi_\tau(z_*) - \pi_\tau(x_i)\| + (\bar{q}+1)\epsilon_{i+1})^2$$

$$= \sum_{\tau \in \mathcal{E}} [\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 + (\bar{q}+1)^2\epsilon_{i+1}^2 + 2(\bar{q}+1)\epsilon_{i+1}\|\pi_\tau(z_*) - \pi_\tau(x_i)\|]$$

$$= \|B_2(z_* - x_i)\|^2 + (\bar{q}+1)^2\epsilon_{i+1}^2 \mathrm{Card}(\mathcal{E})$$

$$+2(\bar{q}+1)\epsilon_{i+1} \sum_{\tau \in \mathcal{E}} \|\pi_\tau(z_*) - \pi_\tau(x_i)\|$$

$$\leq \|B_2(z_* - x_i)\|^2 + (\bar{q}+1)^2\epsilon_{i+1}^2 \mathrm{Card}(\mathcal{E})$$

$$+2(\bar{q}+1)\epsilon_{i+1} \mathrm{Card}(\mathcal{E})\|z_* - x_i\|$$

$$\leq \|B_2(z_* - x_i)\|^2 + (\bar{q}+1)^2\epsilon_{i+1}^2 \mathrm{Card}(\mathcal{E})^2$$

$$+2(\bar{q}+1)\epsilon_{i+1}\|\mathrm{Card}(\mathcal{E})\|B_2(z_* - x_i)\|$$

$$= (\|B_2(z_* - x_i)\| + (\bar{q}+1)\epsilon_{i+1}\mathrm{Card}(\mathcal{E}))^2. \tag{5.87}$$

In view (5.87),

$$\|B_2(z_* - B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau}))\| \leq \|B_2(z_* - x_i)\| + (\bar{q}+1)\epsilon_{i+1}\mathrm{Card}(\mathcal{E}). \tag{5.88}$$

By (5.20), (5.74), and (5.88),

$$\|B_2(z_* - x_{i+1})\|$$

$$\leq \|B_2(z_* - B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau}))\| + \|B_2(B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau})) - x_{i+1}\|$$

$$\leq \|B_2(z_* - x_i)\| + (\bar{q}+1)\epsilon_{i+1}\mathrm{Card}(\mathcal{E}) + \epsilon_{i+1}m_0.$$

Thus

$$\|B_2(z_* - x_{i+1})\| \le \|B_2(z_* - x_i)\| + \epsilon_{i+1}((\bar{q} + 1)\mathrm{Card}(\mathcal{E}) + m_0). \tag{5.89}$$

By induction we show that for all integers $i \ge 0$,

$$\|B_2(z_* - x_i)\| \le 2Mm_0^{1/2} + (\sum_{j=0}^{i} \epsilon_j)((\bar{q} + 1)\mathrm{Card}(\mathcal{E}) + m_0). \tag{5.90}$$

In view of (5.36), (5.70), and (5.82), inequality (5.90) holds for $i = 0$. Assume that $i \ge 0$ is an integer and that (5.90) holds. By (5.89) and (5.90),

$$\|B_2(z_* - x_{i+1})\| \le 2Mm_0^{1/2} + (\sum_{j=0}^{i+1} \epsilon_j)((\bar{q} + 1)\mathrm{Card}(\mathcal{E}) + m_0).$$

Therefore (5.90) holds for all integers $i \ge 0$.

It follows from (5.84), (5.85), and (5.90) that for every integer $i \ge 0$, every $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, every $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and every $j = 0, \dots, p(t)$,

$$\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\| \le \|z_* - x_i\| + \bar{q}\epsilon_{i+1}$$

$$\le 2Mm_0^{1/2} + (\sum_{j=0}^{i+1} \epsilon_j)((\bar{q} + 1)\mathrm{Card}(\mathcal{E}) + m_0), \tag{5.91}$$

$$\|\pi_\tau(z_*) - y_t^{(i,\tau)}\| \le \|z_* - x_i\| + \bar{q}\epsilon_{i+1}$$

$$\le 2Mm_0^{1/2} + (\sum_{j=0}^{i+1} \epsilon_j)((\bar{q} + 1)\mathrm{Card}(\mathcal{E}) + m_0), \tag{5.92}$$

$$\|\pi_\tau(z_*) - y_{i,\tau}\| \le \|z_* - x_i\| + (\bar{q} + 1)\epsilon_{i+1}$$

$$\le 2Mm_0^{1/2} + (\sum_{j=0}^{i+1} \epsilon_j)((\bar{q} + 1)\mathrm{Card}(\mathcal{E}) + m_0). \tag{5.93}$$

It follows from (5.38), (5.71), and (5.90)–(5.93) that for every integer $i \ge 0$, every $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, every $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and every $j = 0, \dots, p(t)$,

$$\|B_2(z_* - x_i)\| \le 2Mm_0^{1/2} + \Lambda((\bar{q} + 1)\mathrm{Card}(\mathcal{E}) + m_0), \tag{5.94}$$

$$\|x_i\| \le 2Mm_0^{1/2} + \Lambda((\bar{q} + 1)\mathrm{Card}(\mathcal{E}) + m_0) + M, \tag{5.95}$$

$$\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\| \le 2Mm_0^{1/2} + \Lambda((\bar{q} + 1)\mathrm{Card}(\mathcal{E}) + m_0), \tag{5.96}$$

$$\|y_{t,j}^{(i,\tau)}\| \le 2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + M, \tag{5.97}$$

$$\|\pi_\tau(z_*) - y_t^{(i,\tau)}\| \le 2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0), \tag{5.98}$$

$$\|y_t^{(i,\tau)}\| \le 2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + M, \tag{5.99}$$

$$\|\pi_\tau(z_*) - y_{i,\tau}\| \le 2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0), \tag{5.100}$$

$$\|y_{i,\tau}\| \le 2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + M. \tag{5.101}$$

Let $i \ge 0$ be an integer,

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}.$$

Assumption (A1), (5.16), and (5.71) imply that for each integer j satisfying $0 \le j < p(t)$,

$$\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|^2$$
$$\ge \bar{c}\|y_{t,j}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|^2. \tag{5.102}$$

It follows from (5.79) and (5.102) that for each integer j satisfying $0 \le j < p(t)$,

$$\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - y_{t,j+1}^{(i,\tau)}\|^2$$
$$\ge \|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|^2$$
$$+\|\pi_\tau(z_*) - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|^2 - -\|\pi_\tau(z_*) - y_{t,j+1}^{(i,\tau)}\|^2$$
$$\ge \bar{c}\|y_{t,j}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|^2$$
$$-\|y_{t,j+1}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|(\|\pi_\tau(z_*) - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|$$
$$+\|\pi_\tau(z_*) - y_{t,j+1}^{(i,\tau)}\|)$$
$$\ge \bar{c}\|y_{t,j}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|^2$$
$$-\epsilon_{i+1}(2\|\pi_\tau(z_*) - y_{t,j+1}^{(i,\tau)}\| + \epsilon_{i+1})$$
$$\ge \bar{c}\|y_{t,j}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|^2$$
$$-\epsilon_{i+1}(4Mm_0^{1/2} + 2\Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + \epsilon_{i+1}). \tag{5.103}$$

In view of (5.79), for each integer j satisfying $0 \le j < p(t)$,

$$\|y_{t,j}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|^2$$

$$\geq \|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|^2 - \|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|^2 + \|y_{t,j}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|^2$$
$$\geq \|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|^2$$
$$-(\|y_{t,j+1}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|(\|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\| + \|y_{t,j}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|)$$
$$\geq \|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|^2 - \epsilon_{i+1}(2\|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\| + \epsilon_{i+1}). \tag{5.104}$$

By (5.39), (5.97), (5.103), and (5.104), for each integer j satisfying $0 \leq j < p(t)$,

$$\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - y_{t,j+1}^{(i,\tau)}\|^2$$
$$\geq \bar{c}\|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|^2$$
$$-\epsilon_{i+1}(2\|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\| + \epsilon_{i+1})$$
$$-\epsilon_{i+1}(4Mm_0^{1/2} + 2\Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + \epsilon_{i+1})$$
$$\geq \bar{c}\|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|^2$$
$$-\epsilon_{i+1}(4Mm_0^{1/2} + 2\Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + 2M + \epsilon_{i+1})$$
$$-\epsilon_{i+1}(4Mm_0^{1/2} + 2\Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + \epsilon_{i+1})$$
$$= \bar{c}\|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|^2$$
$$-\epsilon_{i+1}(8Mm_0^{1/2} + 4\Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + 2M + 2\epsilon_{i+1})$$
$$= \bar{c}\|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|^2 - \epsilon_{i+1}(\tilde{M} + 2\epsilon_{i+1}). \tag{5.105}$$

By (5.27), (5.78), (5.80), (5.81), and (5.105),

$$\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 - \|\pi_\tau(z_*) - y_t^{(i,\tau)}\|^2$$
$$= \|\pi_\tau(z_*) - y_{t,0}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - y_{t,p(t)}^{(i,\tau)}\|^2$$
$$= \sum_{j=0}^{p(t)-1} (\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - y_{t,j+1}^{(i,\tau)}\|^2)$$
$$\geq \sum_{j=0}^{p(t)-1} \bar{c}\|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|^2 - \bar{q}\epsilon_{i+1}(\tilde{M} + 2\epsilon_{i+1})$$
$$\geq \bar{c}(\lambda_t^{(i,\tau)})^2 - \bar{q}\epsilon_{i+1}(\tilde{M} + 2\epsilon_{i+1}). \tag{5.106}$$

It follows from (5.28), (5.29), (5.77), (5.100), (5.106), and the convexity of $\|\cdot\|^2$ that

$$\|\pi_\tau(z_*) - y_{i,\tau}\|^2$$
$$= \|\pi_\tau(z_*) - \sum_{t\in\Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\|^2$$
$$+\|\pi_\tau(z_*) - y_{i,\tau}\|^2 - \|\pi_\tau(z_*) - \sum_{t\in\Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\|^2$$
$$\le \sum_{t\in\Omega_{i+1,\tau}} w_{i+1,\tau}(t)\|\pi_\tau(z_*) - y_t^{(i,\tau)}\|^2$$
$$+\|y_{i,\tau} - \sum_{t\in\Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\|$$
$$\times(\|\pi_\tau(z_*) - y_{i,\tau}\| + \|\pi_\tau(z_*) - \sum_{t\in\Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\|)$$
$$\le \|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2$$
$$+\bar{q}\epsilon_{i+1}(\tilde{M} + 2\epsilon_{i+1}) - \bar{c}\sum_{t\in\Omega_{i+1,\tau}} w_{i+1,\tau}(t)(\lambda_t^{(i,\tau)})^2$$
$$+\epsilon_{i+1}(2\|\pi_\tau(z_*) - y_{i,\tau}\| + \epsilon_{i+1})$$
$$\le \|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 - \bar{c}\lambda_{i,\tau}^2\Delta$$
$$+\bar{q}\epsilon_{i+1}(\tilde{M} + 2\epsilon_{i+1}) + \epsilon_{i+1}(4Mm_0^{1/2} + 2\Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + \epsilon_{i+1})$$
$$\le \|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 - \bar{c}\lambda_{i,\tau}^2\Delta + 2\bar{q}\epsilon_{i+1}(\tilde{M} + 2\epsilon_{i+1}). \tag{5.107}$$

By (5.74) and (5.107),

$$\sum_{\tau\in\mathcal{E}}(\|\pi_\tau(z_*) - y_{i,\tau}\|^2)$$
$$\le \sum_{\tau\in\mathcal{E}}(\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2) - \bar{c}\Delta\sum_{\tau\in\mathcal{E}}\lambda_{i,\tau}^2$$
$$+2\bar{q}\epsilon_{i+1}(\tilde{M} + 2\epsilon_{i+1})\mathrm{Card}(\mathcal{E})$$
$$\le \sum_{\tau\in\mathcal{E}}(\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2) - \bar{c}\Delta\lambda_{i+1}^2 + 2\bar{q}\epsilon_{i+1}(\tilde{M} + 2\epsilon_{i+1})\mathrm{Card}(\mathcal{E}). \tag{5.108}$$

Lemma 5.7 implies that

$$\sum_{\tau\in\mathcal{E}}\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 = \|B_2(z_* - x_i)\|^2. \tag{5.109}$$

Lemma 5.8, (5.20), (5.36), (5.39), (5.74), (5.94), (5.108), and (5.109) imply that

$$\begin{aligned}
&\|B_2(z_* - x_{i+1})\|^2 \\
&= \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau}))\|^2 + \|B_2(z_* - x_{i+1})\|^2 - \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau}))\|^2 \\
&\le \sum_{\tau\in\mathcal{E}} \|\pi_\tau(z_*) - y_{i,\tau}\|^2 + \|B_2(x_{i+1} - B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau}))\| \\
&\quad \times (2\|B_2(z_* - x_{i+1})\| + \|B_2(x_{i+1} - B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau}))\|) \\
&\le \sum_{\tau\in\mathcal{E}} \|\pi_\tau(z_*) - y_{i,\tau}\|^2 + m_0^{1/2}\epsilon_{i+1}(2\|B_2(z_* - x_{i+1})\| + m_0^{1/2}\epsilon_{i+1}) \\
&\le \sum_{\tau\in\mathcal{E}} (\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2) - \bar{c}\Delta\lambda_{i+1}^2 \\
&\quad + 2\bar{q}\epsilon_{i+1}(\tilde{M} + 2\epsilon_{i+1})\mathrm{Card}(\mathcal{E}) \\
&\quad + m_0^{1/2}\epsilon_{i+1}(4Mm_0^{1/2} + 2\Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + m_0^{1/2}\epsilon_{i+1}) \\
&\le \|B_2(z_* - x_i)\|^2 - \bar{c}\Delta\lambda_{i+1}^2 \\
&\quad + 2\bar{q}\epsilon_{i+1}(\tilde{M} + 2\epsilon_{i+1})\mathrm{Card}(\mathcal{E}) \\
&\quad + m_0^{1/2}\epsilon_{i+1}(\tilde{M} + m_0^{1/2}\epsilon_{i+1}).
\end{aligned} \tag{5.110}$$

Set

$$\gamma_0 = \epsilon(\bar{q} + 2)^{-1}. \tag{5.111}$$

In view of (5.40) and (5.111), for all integers $i > n_0$,

$$\epsilon_i < \gamma_0. \tag{5.112}$$

By (5.38), (5.94), and (5.110)–(5.112),

$$\begin{aligned}
&(2Mm_0^{1/2} + 2\Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0))^2 \\
&\ge \|B_2(z_* - x_{n_0})\|^2 \\
&\ge \|B_2(z_* - x_{n_0})\|^2 - \|B_2(z_* - x_n)\|^2 \\
&= \sum_{i=0}^{n-1} (\|B_2(z_* - x_i)\|^2 - \|B_2(z_* - x_{i+1})\|^2)
\end{aligned}$$

$$\geq \bar{c}\Delta \sum_{i=n_0}^{n-1} \lambda_{i+1}^2$$

$$-\sum_{i=n_0}^{n-1} \epsilon_{i+1}[2\bar{q}(\tilde{M} + 2\epsilon_{i+1})\mathrm{Card}(\mathcal{E}) + m_0^{1/2}(\tilde{M} + m_0^{1/2}\epsilon_{i+1})]$$

$$\geq \bar{c}\Delta \sum_{i=0}^{n-1} \lambda_{i+1}^2$$

$$-\sum_{i=n_0}^{n-1} \epsilon_{i+1}[2\bar{q}(\tilde{M} + 1)\mathrm{Card}(\mathcal{E}) + m_0^{1/2}(\tilde{M} + m_0^{1/2})]$$

$$\geq \bar{c}\Delta \sum_{i=n_0}^{n-1} \lambda_{i+1}^2$$

$$-\Lambda[2\bar{q}(\tilde{M} + 1)\mathrm{Card}(\mathcal{E}) + m_0^{1/2}(\tilde{M} + m_0^{1/2})],$$

$$(2Mm_0^{1/2} + \Lambda((\bar{q} + 1)\mathrm{Card}(\mathcal{E}) + m_0))^2$$

$$+\Lambda[2\bar{q}(\tilde{M} + 1)\mathrm{Card}(\mathcal{E}) + m_0^{1/2}(\tilde{M} + m_0^{1/2})]$$

$$\geq \bar{c}\Delta \sum_{i=n_0}^{n-1} \lambda_{i+1}^2$$

$$\geq \bar{c}\Delta\gamma_0^2\mathrm{Card}(\{k \in \{n_0, \dots, n-1\} :\ \lambda_{k+1} \geq \gamma_0\})$$

and

$$\mathrm{Card}(\{k \in \{n_0, \dots .n-1\} :\ \lambda_{k+1} \geq \gamma_0\})$$

$$\leq (\bar{c}\Delta\gamma_0^2)^{-1}[(2Mm_0^{1/2} + \Lambda((\bar{q} + 1)\mathrm{Card}(\mathcal{E}) + m_0))^2$$

$$+\Lambda(2\bar{q}(\tilde{M} + 1)\mathrm{Card}(\mathcal{E}) + m_0^{1/2}(\tilde{M} + m_0^{1/2}))].$$

Since the relation above holds for any natural number $n > n_0$ we conclude that

$$\mathrm{Card}(\{k \in \{n_0, n_0 + 1, \dots, \} :\ \lambda_{k+1} \geq \gamma_0\})$$

$$\leq (\bar{c}\Delta\gamma_0^2)^{-1}[(2Mm_0^{1/2} + \Lambda((\bar{q} + 1)\mathrm{Card}(\mathcal{E}) + m_0))^2$$

$$+\Lambda(2\bar{q}(\tilde{M} + 1)\mathrm{Card}(\mathcal{E}) + m_0^{1/2}(\tilde{M} + m_0^{1/2}))]. \tag{5.113}$$

Assume that an integer $i \geq 0$ satisfies

$$i \geq n_0,\ \lambda_{i+1} < \gamma_0. \tag{5.114}$$

By (5.15), (5.17), (5.74), (5.79), (5.81), (5.112), and (5.114), for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j = 0, \dots, p(t)-1$,

$$\gamma_0 > \lambda_{i+1} \geq \lambda_{i,\tau} \geq \|y_{t,j+1}^{(i,\tau)} - y_{t,j}^{(i,\tau)}\|, \tag{5.115}$$

$$\begin{aligned} &\|y_{t,j}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\| \\ &\leq \|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\| + \|y_{t,j+1}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\| \\ &< \gamma_0 + \epsilon_{i+1} < 2\gamma_0, \end{aligned} \tag{5.116}$$

$$d_{\widehat{X}_\tau}(y_{t,j}^{(i,\tau)}, C_{\tau,t_{j+1}}) < 2\gamma_0. \tag{5.117}$$

Let

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}.$$

In view of (5.27), (5.78), (5.115), and (5.117), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j = 0, 1, \dots, p(t)$,

$$\|\pi_\tau(x_i) - y_{t,j}^{(i,\tau)}\| \leq \gamma_0 j \leq \gamma_0 \bar{q} \tag{5.118}$$

and for each $j = 0, \dots, p(t) - 1$,

$$\begin{aligned} d_{\widehat{X}_\tau}(\pi_\tau(x_i), C_{\tau,t_{j+1}}) &\leq \|\pi_\tau(x_i) - y_{t,j}^{(i,\tau)}\| + d_{\widehat{X}_\tau}(y_{t,j}^{(i,\tau)}, C_{\tau,t_{j+1}}) \\ &< \gamma_0(\bar{q} + 2). \end{aligned} \tag{5.119}$$

By (5.7), (5.8), (5.26), and (5.119), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j = 1, \dots, p(t)$,

$$d_X(x_i, C_{t_j}) \leq \gamma_0(\bar{q} + 2). \tag{5.120}$$

It follows from (5.3), (5.41), (5.111), and (5.120) that

$$d_X(x_i, C_s) \leq \gamma_0(\bar{q} + 2) = \epsilon, \ \ s = 1, \dots, m. \tag{5.121}$$

Thus (5.121) holds for all integers i satisfying (5.114). Together with (5.111) and (5.113) this implies that

$$\begin{aligned} &\mathrm{Card}(\{i \in \{0, 1, \dots\} : \ \max\{d_X(x_i, C_s) : \ s = 1, \dots, m\} > \epsilon\}) \\ &\leq n_0 + \mathrm{Card}(\{i \in \{n_0, n_0 + 1, \dots\} : \ \lambda_{i+1} \geq \gamma_0\}) \\ &\leq n_0 + \bar{c}^{-1} \Delta^{-1} \epsilon^{-2} (\bar{q} + 2)^2 [(2Mm_0^{1/2} + \Lambda((\bar{q} + 1)\mathrm{Card}(\mathcal{E}) + m_0^2)) \\ &+ \Lambda(2\bar{q}(\tilde{M} + 1)\mathrm{Card}(\mathcal{E}) + m_0^{1/2}(\tilde{M} + m_0^{1/2}))]. \end{aligned}$$

Theorem 5.1 is proved.

5.4 Auxiliary Results for Theorems 5.2, 5.3, 5.5, and 5.6

We suppose that for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$ and each $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$,

$$\|P_{\tau,s}(x_1) - P_{\tau,s}(x_2)\| \le \|x_1 - x_2\| \tag{5.122}$$

for all $x_1, x_2 \in \widehat{X}_\tau$.

By (5.22) and (5.122), for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each index vector $t = (t_1, \dots, t_{p(t)})$ such that

$$t_i \in \{\tau_1, \dots, \tau_{p(\tau)}\},\ i = 1, \dots, p(t),$$

we have

$$\|P[t](x_1) - P[t](x_2)\| \le \|x_1 - x_2\| \tag{5.123}$$

for all $x_1, x_2 \in \widehat{X}_\tau$. By (5.30), (5.123) and the convexity of the norm, for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$ and each $(\Omega, w) \in \mathcal{M}_\tau$, we have

$$\|P_{\Omega,w}(x_1) - P_{\Omega,w}(x_2)\| \le \|x_1 - x_2\| \tag{5.124}$$

for all $x_1, x_2 \in \widehat{X}_\tau$.

Let

$$x, y \in X,\ (\Omega_\tau, w_\tau) \in \mathcal{M}_\tau,\ \tau \in \mathcal{E}.$$

Lemmas 5.7 and 5.8, (5.20), and (5.124) imply that

$$\begin{aligned}
&\|B_2 \circ B_1(\sum_{\tau\in\mathcal{E}} P_{\Omega_\tau,w_\tau}(\pi_\tau(x))) - B_2 \circ B_1(\sum_{\tau\in\mathcal{E}} P_{\Omega_\tau,w_\tau}(\pi_\tau(y)))\|^2 \\
&= \|B_2 \circ B_1(\sum_{\tau\in\mathcal{E}} P_{\Omega_\tau,w_\tau}(\pi_\tau(x)) - \sum_{\tau\in\mathcal{E}} P_{\Omega_\tau,w_\tau}(\pi_\tau(y)))\|^2 \\
&= \|B_2 \circ B_1(\sum_{\tau\in\mathcal{E}}(P_{\Omega_\tau,w_\tau}(\pi_\tau(x)) - P_{\Omega_\tau,w_\tau}(\pi_\tau(y))))\|^2 \\
&\le \sum_{\tau\in\mathcal{E}} \|P_{\Omega_\tau,w_\tau}(\pi_\tau(x)) - P_{\Omega_\tau,w_\tau}(\pi_\tau(y))\|^2 \\
&\le \sum_{\tau\in\mathcal{E}} \|\pi_\tau(x) - \pi_\tau(y)\|^2 = \|B_2(x-y)\|^2.
\end{aligned} \tag{5.125}$$

Lemma 5.9 *Let* $x, y \in X$, $\lambda, \delta \ge 0$,

$$(\Omega_\tau, w_\tau) \in \mathcal{M}_\tau,\ \tau \in \mathcal{E}$$

and

$$(y, \lambda) \in A(x, \{(\Omega_\tau, w_\tau)\}_{\tau \in \mathcal{E}}, \delta). \tag{5.126}$$

Then

$$\|B_2(y - B_1(\sum_{\tau \in \mathcal{E}} P_{\Omega_\tau, w_\tau}(\pi_\tau(x))))\| \le m_0^{1/2}\delta(\bar{q} + 1)Card(\mathcal{E}).$$

Proof In view of (5.35) and (5.126), for each $\tau \in \mathcal{E}$, there exist

$$(y_\tau, \lambda_\tau) \in A_\tau(\pi_\tau(x), (\Omega_\tau, w_\tau), \delta), \ \ \tau \in \mathcal{E} \tag{5.127}$$

such that

$$\|y - B_1(\sum_{\tau \in \mathcal{E}} y_\tau)\| \le \delta \tag{5.128}$$

$$\lambda = \max\{\lambda_\tau \ : \ \tau \in \mathcal{E}\}. \tag{5.129}$$

By (5.34) and (5.127), for each $\tau \in \mathcal{E}$ there exist

$$(y_t^{(\tau)}, \lambda_t^{(\tau)}) \in A_{\tau,0}(\pi_\tau(x), t, \delta), \ \ t \in \Omega_\tau \tag{5.130}$$

such that

$$\|y_\tau - \sum_{t \in \Omega_\tau} w_\tau(t) y_t^{(\tau)}\| \le \delta, \tag{5.131}$$

$$\lambda_\tau = \max\{\lambda_t^{(\tau)} \ : \ t \in \Omega_\tau\}. \tag{5.132}$$

By (5.33) and (5.130), for each $\tau \in \mathcal{E}$ and each $t = (t_1, \dots, t_{p(t)}) \in \Omega_\tau$ there exists a sequence

$$\{y_{t,j}^{(\tau)}\}_{j=0}^{p(t)} \subset \widehat{X}_\tau$$

such that

$$y_{t,0}^{(\tau)} = \pi_\tau(x), \tag{5.133}$$

for all $j = 1, \dots, p(t)$,

$$\|y_{t,j}^{(\tau)} - P_{\tau,t_j}(y_{t,j-1}^{(\tau)})\| \le \delta, \tag{5.134}$$

$$y_t^{(\tau)} = y_{t,p(t)}^{(\tau)}, \tag{5.135}$$

$$\lambda_t^{(\tau)} = \max\{\|y_{t,j}^{(\tau)} - y_{t,j-1}^{(\tau)}\| : \ j = 1, \dots, p(t)\}. \tag{5.136}$$

In view of (5.30), for every $\tau \in \mathcal{E}$,

$$P_{\Omega_\tau, w_\tau}(\pi_\tau(x)) = \sum_{t \in \Omega_\tau} w_\tau(t) P[t](\pi_\tau(x)) \tag{5.137}$$

and by (5.22), for every $\tau \in \mathcal{E}$ and every $t = (t_1, \dots, t_{p(t)}) \in \Omega_\tau$,

$$P[t](\pi_\tau(x)) = P_{\tau, t_{p(t)}} \cdots P_{\tau, t_{(1)}}(\pi_\tau(x)). \tag{5.138}$$

Proposition 2.8, (5.27), (5.122), (5.133)–(5.135), and (5.138) imply that for every $t = (t_1, \dots, t_{p(t)}) \in \Omega_\tau$,

$$\|y_t^{(\tau)} - P[t](\pi_\tau(x))\| \le p(t)\delta \le \bar{q}\delta. \tag{5.139}$$

By (5.28), (5.131), (5.137), and the convexity of the norm, for every $\tau \in \mathcal{E}$,

$$\begin{aligned}
&\|y_\tau - P_{\Omega_\tau, w_\tau}(\pi_\tau(x))\| \\
&\le \|y_\tau - \sum_{t \in \Omega_\tau} w_\tau(t) y_t^{(\tau)}\| \\
&+\|\sum_{t \in \Omega_\tau} w_\tau(t) y_t^{(\tau)} - \sum_{t \in \Omega_\tau} w_\tau(t) P[t](\pi_\tau(x))\| \\
&\le \delta + \sum_{t \in \Omega_\tau} w_\tau(t) \|y_t^{(\tau)} - P[t](\pi_\tau(x))\| \le \delta + \bar{q}\delta.
\end{aligned} \tag{5.140}$$

In view of (5.20) and (5.128),

$$\begin{aligned}
&\|y - B_1(\sum_{\tau \in \mathcal{E}} P_{\Omega_\tau, w_\tau}(\pi_\tau(x)))\| \\
&\le \|y - B_1(\sum_{\tau \in \mathcal{E}} y_\tau)\| \\
&+\|B_1(\sum_{\tau \in \mathcal{E}} y_\tau - \sum_{\tau \in \mathcal{E}} P_{\Omega_\tau, w_\tau}(\pi_\tau(x)))\| \\
&\le \delta(\bar{q} + 2)\mathrm{Card}\mathcal{E}
\end{aligned}$$

and

$$\|B_2(y - B_1(\sum_{\tau \in \mathcal{E}} P_{\Omega_\tau, w_\tau}(\pi_\tau(x))))\| \le m_0^{1/2}\delta(\bar{q} + 2)\mathrm{Card}\mathcal{E}.$$

Lemma 5.9 is proved.

5.5 Proof of Theorem 5.2

Set

$$\gamma_0 = \epsilon^2(\bar{q}+1)^{-2}(4Mm_0\mathrm{Card}(\mathcal{E})^{1/2}\bar{N}\bar{q})^{-1}\Delta\bar{c}. \tag{5.141}$$

In view of (5.45), there exists

$$z_* = (z_{*,1}, \dots, z_{*,l}) \in B_X(0, M) \cap C. \tag{5.142}$$

It follows from (5.44), (5.46), (5.123), and (5.124) that for every integer $i \geq 0$, every $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, every $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$,

$$\|P[t](x) - P[t](y)\| \leq \|x - y\| \tag{5.143}$$

for all $x, y \in \widehat{X}_\tau$ and

$$\|P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(x) - P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(y)\| \leq \|x - y\| \tag{5.144}$$

for all $x, y \in \widehat{X}_\tau$.

Let $i \geq 0$ be an integer. By (5.49),

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, \{(\Omega_{i+1,\tau}, w_{i+1,\tau})\}_{\tau\in\mathcal{E}}, 0). \tag{5.145}$$

In view of (5.35) and (5.145), there exist

$$(y_{i,\tau}, \lambda_{i,\tau}) \in A_\tau(\pi_\tau(x_i), (\Omega_{i+1,\tau}, w_{i+1,\tau}), 0),\ \ \tau \in \mathcal{E} \tag{5.146}$$

such that

$$x_{i+1} = B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau}),\ \ \lambda_{i+1} = \max\{\lambda_{i,\tau} :\ \tau \in \mathcal{E}\}. \tag{5.147}$$

By (5.34) and (5.146), for each $\tau \in \mathcal{E}$, there exist

$$(y_t^{(i,\tau)}, \lambda_t^{(i,\tau)}) \in A_{\tau,0}(\pi_\tau(x_i), t, 0),\ \ t \in \Omega_{i+1,\tau} \tag{5.148}$$

such that

$$y_{i,\tau} = \sum_{t\in\Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}, \tag{5.149}$$

$$\lambda_{i,\tau} = \max\{\lambda_t^{(i,\tau)} :\ t \in \Omega_{i+1,\tau}\}. \tag{5.150}$$

By (5.33) and (5.148), for each $\tau \in \mathcal{E}$ and each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ there exists a sequence

$$\{y_{t,j}^{(i,\tau)}\}_{j=0}^{p(t)} \subset \widehat{X}_\tau$$

such that

$$y_{t,0}^{(i,\tau)} = \pi_\tau(x_i), \tag{5.151}$$

for all $j = 1, \dots, p(t)$,

$$y_{t,j}^{(i,\tau)} = P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)}), \tag{5.152}$$

$$y_t^{(i,\tau)} = y_{t,p(t)}^{(i,\tau)}, \tag{5.153}$$

$$\lambda_t^{(i,\tau)} = \max\{\|y_{t,j}^{(i,\tau)} - y_{t,j-1}^{(i,\tau)}\| : \ j = 1, \dots, p(t)\}. \tag{5.154}$$

It follows from (5.30), (5.147), (5.149), and (5.151)–(5.153) that for each $\tau \in \mathcal{E}$ and each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$,

$$\begin{aligned} y_t^{(i,\tau)} &= P_{\tau,t_{p(t)}} \cdots P_{\tau,t_{(1)}}(y_0^{(i,\tau)}) \\ &= P_{\tau,t_{p(t)}} \cdots P_{\tau,t_{(1)}}(\pi_\tau(x_i)) = P[t](\pi_\tau(x_i)), \end{aligned} \tag{5.155}$$

$$y_{i,\tau} = \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) P[t](\pi_\tau(x_i)) = P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(\pi_\tau(x_i)), \tag{5.156}$$

$$x_{i+1} = B_1\Big(\sum_{\tau \in \mathcal{E}} P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(\pi_\tau(x_i))\Big), \tag{5.157}$$

$$B_2(x_{i+1}) = B_2 \circ B_1\Big(\sum_{\tau \in \mathcal{E}} P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(\pi_\tau(x_i))\Big). \tag{5.158}$$

By (5.8), (5.12), (5.20), (5.22), (5.30), and (5.142), for each $\tau \in \mathcal{E}$ and each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$,

$$P[t](\pi_\tau(z_*)) = \pi_\tau(z_*),$$

$$P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(\pi_\tau(z_*)) = \pi_\tau(z_*)$$

and

$$B_1\Big(\sum_{\tau \in \mathcal{E}} P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(\pi_\tau(z_*))\Big) = B_1\Big(\sum_{\tau \in \mathcal{E}} \pi_\tau(z_*)\Big) = z_*. \tag{5.159}$$

Relations (5.125), (5.158), and (5.159) imply that

$$\|B_2(z_* - x_{i+1})\| \leq \|B_2(z_* - x_i)\|. \tag{5.160}$$

Let

$$\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \mathcal{E},\ t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1,\tau}.$$

By (5.13), (5.142), (5.152), and (5.153), for each integer j satisfying $0 \leq j < p(t)$,

$$\begin{aligned} &\|\pi_\tau(z_*) - y^{(i,\tau)}_{t,j+1}\| \\ &= \|\pi_\tau(z_*) - P_{\tau,t_{j+1}}(y^{(i,\tau)}_{t,j})\| \\ &\leq \|\pi_\tau(z_*) - y^{(i,\tau)}_{t,j}\|. \end{aligned} \tag{5.161}$$

It follows from (5.151) and (5.161) that for all integers $j \in \{0, \ldots, p(t)\}$,

$$\begin{aligned} \|\pi_\tau(z_*) - y^{(i,\tau)}_{t,j}\| &\leq \|\pi_\tau(z_*) - y^{(i,\tau)}_{t,0}\| \\ &= \|\pi_\tau(z_*) - \pi_\tau(x_i)\| \end{aligned} \tag{5.162}$$

and

$$\|\pi_\tau(z_*) - y^{(i,\tau)}_{t}\| \leq \|\pi_\tau(z_*) - \pi_\tau(x_i)\|. \tag{5.163}$$

By (5.28), (5.149), (5.163), and the convexity of the norm,

$$\begin{aligned} &\|\pi_\tau(z_*) - y_{i,\tau}\| \\ &= \|\pi_\tau(z_*) - \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) y^{(i,\tau)}_{t}\| \\ &\leq \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) \|\pi_\tau(z_*) - y^{(i,\tau)}_{t}\| \\ &\leq \|\pi_\tau(z_*) - \pi_\tau(x_i)\|. \end{aligned} \tag{5.164}$$

In view of (5.20), (5.48), (5.142), and (5.160), for all integers $i \geq 0$,

$$\|x_i - z_*\| \leq \|B_2(z_* - x_0)\| \leq m_0^{1/2}\|x_0 - z_*\| \leq 2m_0^{1/2}M, \tag{5.165}$$

$$\|x_i\| \leq 2m_0^{1/2}M + M. \tag{5.166}$$

By (5.162), (5.164), and (5.165), for each integer $i \geq 0$, for each $\tau \in \mathcal{E}$, each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1,\tau}$, and each $j = 0, \ldots, p(t)$,

$$\|\pi_\tau(z_*) - y^{(i,\tau)}_{t,j}\| \leq 2M + m_0^{1/2}, \tag{5.167}$$

$$\|\pi_\tau(z_*) - y_t^{(i,\tau)}\| \le 2M + m_0^{1/2}, \tag{5.168}$$

$$\|\pi_\tau(z_*) - y_{i,\tau}\| \le 2M + m_0^{1/2}. \tag{5.169}$$

For each integer $i \ge 0$ and each $x \in X$, set

$$T_{i+1}(x) = B_1(\sum_{\tau \in \mathcal{E}} P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(\pi_\tau(x_i))). \tag{5.170}$$

In view of (5.47) and (5.170), for each integer $i \ge 1$,

$$T_{i+\bar{N}} = T_i. \tag{5.171}$$

Relations (5.157) and (5.170) imply that for each integer $i \ge 0$,

$$x_{i+1} = T_{i+1}(x_i). \tag{5.172}$$

It follows from (5.125) and (5.170) that for all $x, y \in X$,

$$\|B_2(T_i(x)) - B_2(T_i(y))\| \le \|B_2(x - y)\|. \tag{5.173}$$

Assumption (A1), (5.7), (5.8), (5.16), (5.142), and (5.152) imply that for each integer $i \ge 0$, each $\tau \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$, and each $j = 0, \dots, p(t) - 1$,

$$\begin{aligned} &\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - y_{t,j+1}^{(i,\tau)}\|^2 \\ &= \|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|^2 \\ &\ge \bar{c}\|y_{t,j}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|^2. \end{aligned} \tag{5.174}$$

By (5.151)–(5.154) and (5.174), for each integer $i \ge 0$, for each $\tau \in \mathcal{E}$, and each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$,

$$\begin{aligned} &\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 - \|\pi_\tau(z_*) - y_t^{(i,\tau)}\|^2 \\ &= \|\pi_\tau(z_*) - y_{t,0}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - y_{t,p(t)}^{(i,\tau)}\|^2 \\ &= \sum_{j=0}^{p(t)-1} (\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - y_{t,j+1}^{(i,\tau)}\|^2) \\ &\ge \sum_{j=0}^{p(t)-1} \bar{c}\|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|^2 \\ &\ge \bar{c}(\lambda_t^{(i,\tau)})^2. \end{aligned} \tag{5.175}$$

It follows from (5.28), (5.29), (5.149), (5.150), (5.175), and the convexity of $\|\cdot\|^2$ that for each integer $i \geq 0$ and each $\tau \in \mathcal{E}$,

$$\begin{aligned}
&\|\pi_\tau(z_*) - y_{i,\tau}\|^2 \\
&= \|\pi_\tau(z_*) - \sum_{t\in\Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\|^2 \\
&\leq \sum_{t\in\Omega_{i+1,\tau}} w_{i+1,\tau}(t)\|\pi_\tau(z_*) - y_t^{(i,\tau)}\|^2 \\
&\leq \|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 - \bar{c} \sum_{t\in\Omega_{i+1,\tau}} w_{i+1,\tau}(t)(\lambda_t^{(i,\tau)})^2 \\
&\leq \|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 - \bar{c}\lambda_{i,\tau}^2 \Delta.
\end{aligned} \tag{5.176}$$

By Lemmas 5.7 and 5.8, (5.147), (5.156), (5.157), and (5.176), for each integer $i \geq 0$,

$$\begin{aligned}
&\|B_2(z_* - x_{i+1})\|^2 \\
&= \|B_2 \circ B_1(\sum_{\tau\in\mathcal{E}} P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(\pi_\tau(x_i))) - B_2(z_*)\|^2 \\
&= \|B_2(B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau})) - B_2(z_*)\|^2 \\
&\leq \sum_{\tau\in\mathcal{E}} \|y_{i,\tau} - \pi_\tau(z_*)\|^2 \\
&\leq \sum_{\tau\in\mathcal{E}} \|\pi_\tau(x_i) - \pi_\tau(z_*)\|^2 - \sum_{\tau\in\mathcal{E}} \bar{c}\lambda_{i,\tau}^2 \Delta \\
&\leq \|B_2(z_* - x_i)\|^2 - \bar{c}\Delta\lambda_{i+1}^2.
\end{aligned} \tag{5.177}$$

Let n be a natural number. By (5.165) and (5.177),

$$\begin{aligned}
&4M^2 m_0 \geq \|B_2(z_* - x_0)\|^2 \\
&\geq \|B_2(z_* - x_0)\|^2 - \|B_2(z_* - x_{\bar{N}n})\|^2 \\
&= \sum_{k=0}^{n-1}(\|B_2(z_* - x_{k\bar{N}})\|^2 - \|B_2(z_* - x_{(k+1)\bar{N}})\|^2) \\
&= \sum_{k=0}^{n-1}(\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} (\|B_2(z_* - x_j)\|^2 - \|B_2(z_* - x_{j+1})\|^2))
\end{aligned}$$

$$\geq \sum_{k=0}^{n-1} \sum_{j=k\bar{N}}^{(k+1)\bar{N}-1} \geq \bar{c}\Delta\lambda_{j+1}^2$$

$$\geq \bar{c}\Delta\gamma_0^2 \mathrm{Card}(\{k \in \{0, \dots, n-1\} :$$

$$\max\{\lambda_{i+1} : \ i = k\bar{N}, \dots, (k+1)\bar{N} - 1\} \geq \gamma_0\})$$

and

$$\mathrm{Card}(\{k \in \{0, \dots, n-1\} :$$

$$\max\{\lambda_{i+1} : \ i = k\bar{N}, \dots, (k+1)\bar{N} - 1\} \geq \gamma_0\})$$

$$\leq 4M^2 m_0 (\bar{c}\Delta\gamma_0^2)^{-1}.$$

Since n is any natural number the relation above implies that

$$\mathrm{Card}(\{k \in \{0, 1, \dots\} : \ \max\{\lambda_{i+1} :$$

$$i = k\bar{N}, \dots, (k+1)\bar{N} - 1\} \geq \gamma_0\})$$

$$\leq 4M^2 m_0 (\bar{c}\Delta\gamma_0^2)^{-1}. \tag{5.178}$$

In view of (5.141) and (5.178), there exists an integer $q_0 \geq 0$ such that

$$q_0 \leq 4M^2 m_0 \bar{c}^{-1}\Delta^{-1}\gamma_0^{-2} + 1$$

$$= 1 + 4M^2 m_0 \bar{c}^{-1}\Delta^{-1}\epsilon^{-4}(\bar{q}+1)^4\Delta^{-2}\bar{c}^{-2}(4Mm_0\mathrm{Card}(\mathcal{E})^{1/2}\bar{N}\bar{q})^2 \tag{5.179}$$

and

$$\lambda_{i+1} < \gamma_0, \ i = q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1. \tag{5.180}$$

By (5.27), (5.147), (5.150), (5.151), (5.153), (5.154), and (5.180), for each integer $i \in \{q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1\}$, for each $\tau \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$, and each $j = 0, \dots, p(t) - 1$,

$$\gamma_0 > \lambda_{i+1} \geq \lambda_{i,\tau} \geq \lambda_t^{(i,\tau)} \geq \|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|, \tag{5.181}$$

$$\|\pi_\tau(x_i) - y_{t,j+1}^{(i,\tau)}\| \leq (j+1)\gamma_0, \tag{5.182}$$

$$\|\pi_\tau(x_i) - y_t^{(i,\tau)}\| \leq p(t)\gamma_0 \leq \bar{q}\gamma_0. \tag{5.183}$$

By (5.28), (5.149), (5.183), and the convexity of the norm, for each $i \in \{q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1\}$ and each $\tau \in \mathcal{E}$,

$$\|\pi_\tau(x_i) - y_{i,\tau}\|$$
$$= \|\pi_\tau(x_i) - \sum_{t\in\Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\|$$
$$\le \sum_{t\in\Omega_{i+1,\tau}} w_{i+1,\tau}(t)\|\pi_\tau(x_i) - y_t^{(i,\tau)}\| \le \bar{q}\gamma_0. \tag{5.184}$$

Lemma 5.8, (5.147), and (5.184) imply that for each $i \in \{q_0\bar{N}, \ldots, (q_0+1)\bar{N}-1\}$,

$$\|B_2(x_i - x_{i+1})\|^2$$
$$= \|B_2(x_i - B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau}))\|^2$$
$$\le \sum_{\tau\in\mathcal{E}} \|\pi_\tau(x_i) - y_{i,\tau}\|^2 \le \mathrm{Card}(\mathcal{E})\bar{q}^{-2}\gamma_0^2$$

and

$$\|x_i - x_{i+1}\| \le \mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0. \tag{5.185}$$

In view of (5.185),

$$\|x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}}\| \le \bar{N}\mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0. \tag{5.186}$$

By (5.171)–(5.173) and (5.186), for each integer $q > q_0$,

$$\|B_2(x_{q\bar{N}} - x_{(q+1)\bar{N}})\|$$
$$= \|B_2(\prod_{j=q_0\bar{N}+1}^{q\bar{N}} T_j(x_{q_0\bar{N}}) - \prod_{j=(q_0+1)\bar{N}+1}^{(q+1)\bar{N}} T_j(x_{(q_0+1)\bar{N}}))\|$$
$$= \|B_2(\prod_{j=q_0\bar{N}+1}^{q\bar{N}} T_j(x_{q_0\bar{N}}) - \prod_{j=q_0\bar{N}+1}^{q\bar{N}} T_j(x_{(q_0+1)\bar{N}}))\|$$
$$\le \|B_2(x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}})\|. \tag{5.187}$$

In view of (5.20), (5.186), and (5.187), for all integers $q \ge q_0$.

$$\|B_2(x_{q\bar{N}} - x_{(q+1)\bar{N}})\| \le \|B_2(x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}})\|$$
$$\le m_0^{1/2}\bar{N}\mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0. \tag{5.188}$$

Let $q \ge q_0$ be an integer. It follows from (5.160), (5.165), (5.177), and (5.188),

$$m_0^{1/2}\bar{N}\mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0$$
$$\ge \|B_2(x_{q\bar{N}} - x_{(q+1)\bar{N}})\|$$
$$\ge \|B_2(z_* - x_{q\bar{N}})\| - \|B_2(z_* - x_{(q+1)\bar{N}})\|$$
$$\ge (\|B_2(z_* - x_{q\bar{N}})\|^2 - \|B_2(z_* - x_{(q+1)\bar{N}})\|^2)(4Mm_0^{1/2})^{-1}$$

and

$$4Mm_0\bar{N}\mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0$$
$$\ge \|B_2(z_* - x_{q\bar{N}})\|^2 - \|B_2(z_* - x_{(q+1)\bar{N}})\|^2$$
$$= \sum_{i=q\bar{N}}^{(q+1)\bar{N}-1} [\|B_2(z_* - x_i)\|^2 - \|B_2(z_* - x_{i+1})\|^2]$$
$$\ge \sum_{i=q\bar{N}}^{(q+1)\bar{N}-1} \Delta\bar{c}\lambda_{i+1}^2$$

and

$$\lambda_{i+1} \le (4Mm_0\bar{N}\mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0\Delta^{-1}\bar{c}^{-1})^{1/2} \tag{5.189}$$

for all $i = q\bar{N}, \dots, (q+1)\bar{N} - 1$. Set

$$\Gamma = (4Mm_0\bar{N}\mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0\Delta^{-1}\bar{c}^{-1})^{1/2}.$$

By (5.27), (5.147), (5.150)–(5.152), (5.154), and (5.189), for each integer $i \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$, each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$, and each $j = 0, \dots, p(t) - 1$,

$$\Gamma = (4Mm_0\bar{N}\mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0\Delta^{-1}\bar{c}^{-1})^{1/2}$$
$$\ge \lambda_{i+1} \ge \lambda_{i,\tau} \ge \lambda_t^{(i,\tau)}$$
$$\ge \|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\| = \|y_{t,j}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|$$
$$\ge d_{\widehat{X}_\tau}(y_{t,j}^{(i,\tau)}, C_{\tau,t_{j+1}}),$$
$$\|\pi_\tau(x_i) - y_{t,j}^{(i,\tau)}\| \le j\Gamma \le \bar{q}\Gamma$$

and

$$d_{\widehat{X}_\tau}(\pi_\tau(x_i), C_{\tau,t_{j+1}}) \le \|\pi_\tau(x_i) - y_{t,j}^{(i,\tau)}\| + d_{\widehat{X}_\tau}(y_{t,j}^{(i,\tau)}, C_{\tau,t_{j+1}}) \le (\bar{q}+1)\Gamma. \tag{5.190}$$

By (5.3), (5.7), (5.8), (5.25), (5.26), (5.46), and (5.190), for each integer $i \in \{q\bar{N}, \dots, (q+1)\bar{N}-1\}$, each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$, and each $j = 1, \dots, p(t)$,

$$d_X(x_i, C_{t_j}) = d_{\widehat{X}_\tau}(\pi_\tau(x_i), C_{\tau,t_j}) \le (\bar{q}+1)\Gamma$$

and

$$d_X(x_i, C_s) \le (\bar{q}+1)\Gamma$$

for all $s = 1, \dots, m$. By the relation above, (5.141) and (5.190), for all integers $i \ge q_0\bar{N}$ and all $s = 1, \dots, m$,

$$d_X(x_i, C_s) \le (\bar{q}+1)(4Mm_0\bar{N}\mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0\Delta^{-1}\bar{c}^{-1})^{1/2} \le \epsilon.$$

Theorem 5.2 is proved.

5.6 Proof of Theorem 5.3

Theorem 5.3 is deduced from Theorems 2.9 and 5.2. Let $Y = X$, $N = \bar{N}$, $\rho(x, y) = \|B_2(x-y)\|$, $x, y \in X$, $\mathfrak{A}$ be the set of all mappings S defined on the set of natural numbers into the set of operators

$$B_1 \circ (\sum_{\tau\in\mathcal{E}} P_{\Omega_\tau, w_\tau} \circ \pi_\tau) : X \to X,$$

with

$$(\Omega_\tau, w_\tau) \in \mathcal{M}_\tau,\ \ \tau \in \mathcal{E}$$

such that for each integer $i \ge 1$,

$$S(i) = B_1 \circ (\sum_{\tau\in\mathcal{E}} P_{\Omega_{\tau(i)}, w_{\tau(i)}} \circ \pi_\tau),$$

where

$$(\Omega_{\tau(i)}, w_{\tau(i)}) \in \mathcal{M}_\tau,\ \ \tau \in \mathcal{E}$$

satisfy

$$(\Omega_{\tau(i+\bar{N})}, w_{\tau(i+\bar{N})}) = (\Omega_{\tau(i)}, w_{\tau(i)})$$

for all integers $i \geq 1$. Let

$$F = \{x \in X : \ d_X(x, C_s) \leq \epsilon_0/4, \ s = 1, \dots, m\}. \tag{5.191}$$

Theorem 5.2 implies that for every $M > 0$ there exists $Q > 0$ such that property (P6) holds. Lemma 5.9 and (5.54) imply that for each integer $i \geq 0$,

$$\|B_2(x_{i+1} - B_1(\sum_{\tau\in\mathcal{E}} P_{\Omega_{i+1,\tau}, w_{i+1,\tau}} \circ \pi_\tau)(x_i))\|$$

$$\leq m_0^{1/2}\delta(\bar{q}+1)\mathrm{Card}(\mathcal{E}) \leq 4^{-1}\epsilon_0 Q^{-1}(2\bar{N}+1)^{-1}. \tag{5.192}$$

By Theorems 2.9 and 5.2, (5.53), and (5.192), for all integer $i \geq Q$,

$$B(x_i, \epsilon_0/4) \cap F \neq \emptyset.$$

Together with (5.191) this implies that for all integers $i \geq Q$,

$$d_X(x_i, C_s) < \epsilon_0, \ s = 1, \dots, m.$$

Theorem 5.3 is proved.

5.7 Proof of Theorem 5.4

By (5.59), there exists

$$z_* = (z_{*,1}, \dots, z_{*,l}) \in B_X(0, M) \cap C. \tag{5.193}$$

Set

$$\gamma_0 = \epsilon(\bar{q}+2)^{-1}, \tag{5.194}$$

$$M_1 = 2(\bar{q} + m_0^{1/2})(2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0 + 1)\mathrm{Card}(\mathcal{E}), \tag{5.195}$$

$$M_2 = ((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0)2(2M + m_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0). \tag{5.196}$$

By (A2) there exists $\gamma \in (0, \gamma_0)$ such that the following property holds:

(P1) for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$, each $z \in C_{\tau,s}$ satisfying $\|z\| \le M$, and each $x \in \widehat{X}_\tau$ satisfying

$$\|x\| \le 2Mm_0 + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + M$$

and

$$d_{\widehat{X}_\tau}(x, C_{\tau,s}) \ge \gamma_0/4$$

the inequality

$$\|z - P_{\tau,s}(x)\| \le \|z - x\| - \gamma$$

holds.

In view of (5.60), there exists a natural number n_0 such that for each integer $i > n_0$,

$$\epsilon_i < (\gamma/4)(\bar{q}+1)^{-1}. \tag{5.197}$$

Set

$$Q = n_0 + 4\gamma^{-2}\Delta^{-2}[\Lambda(M_1 + M_2) + (2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0 + M)^2]. \tag{5.198}$$

Assume that for all natural numbers i,

$$(\Omega_{i,\tau}, w_{i,\tau}) \in \mathcal{M}_\tau,\ \ \tau \in \mathcal{E}, \tag{5.199}$$

$$x_0 \in B_X(0, M), \tag{5.200}$$

$\{x_i\}_{i=1}^\infty \subset X,\ \{\lambda_i\}_{i=1}^\infty \subset [0, \infty)$ and that for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau \in \mathcal{E}}, \epsilon_i). \tag{5.201}$$

Let $i \ge 0$ be an integer. By (5.201),

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, \{(\Omega_{i+1,\tau}, w_{i+1,\tau})\}_{\tau \in \mathcal{E}}, \epsilon_{i+1}). \tag{5.202}$$

In view of (5.35) and (5.202), there exist

$$(y_{i,\tau}, \lambda_{i,\tau}) \in A_\tau(\pi_\tau(x_i), (\Omega_{i+1,\tau}, w_{i+1,\tau}), \epsilon_{i+1}),\ \ \tau \in \mathcal{E} \tag{5.203}$$

such that

$$\|x_{i+1} - B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau})\| \le \epsilon_{i+1}, \tag{5.204}$$

$$\lambda_{i+1} = \max\{\lambda_{i,\tau} : \tau \in \mathcal{E}\}. \tag{5.205}$$

By (5.34) and (5.203), for each $\tau \in \mathcal{E}$ there exist

$$(y_t^{(i,\tau)}, \lambda_t^{(i,\tau)}) \in A_{\tau,0}(\pi_\tau(x_i), t, \epsilon_{i+1}), \; t \in \Omega_{i+1,\tau} \tag{5.206}$$

such that

$$\|y_{i,\tau} - \sum_{t\in\Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\| \le \epsilon_{i+1}, \tag{5.207}$$

$$\lambda_{i,\tau} = \max\{\lambda_t^{(i,\tau)} : t \in \Omega_{i+1,\tau}\}. \tag{5.208}$$

By (5.33) and (5.206), for each $\tau \in \mathcal{E}$ and each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1,\tau}$ there exists a sequence

$$\{y_{t,j}^{(i,\tau)}\}_{j=0}^{p(t)} \subset \widehat{X}_\tau$$

such that

$$y_{t,0}^{(i,\tau)} = \pi_\tau(x_i), \tag{5.209}$$

for all $j = 1, \ldots, p(t)$,

$$\|y_{t,j}^{(i,\tau)} - P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)})\| \le \epsilon_{i+1}, \tag{5.210}$$

$$y_t^{(i,\tau)} = y_{t,p(t)}^{(i,\tau)}, \tag{5.211}$$

$$\lambda_t^{(i,\tau)} = \max\{\|y_{t,j}^{(i,\tau)} - y_{t,j-1}^{(i,\tau)}\| : j = 1, \ldots, p(t)\}. \tag{5.212}$$

In view of (5.20), (5.36), (5.193), and (5.200),

$$\|z_* - x_0\| \le 2M \tag{5.213}$$

and

$$\|B_2(z_* - x_0)\| \le 2Mm_0^{1/2}. \tag{5.214}$$

Let

$$\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \mathcal{E}, \; t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1,\tau}.$$

By (5.12), (5.13), (5.27), (5.193), and (5.209)–(5.211), arguing as in Section 5.3 we show that for each integer j satisfying $0 \le j < p(t)$, Equation (5.83) holds, for all integers $j \in \{0, \ldots, p(t)\}$, Equations (5.84) and (5.85) hold and by the convexity of the norm, (5.27), (5.28), and (5.85), Equation (5.86) holds.

Thus for all integers $j \in \{0, \dots, p(t)\}$,

$$\text{if } j < p(t), \text{ then } \|\pi_\tau(z_*) - y_{t,j+1}^{(i,\tau)}\| \le \|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\| + \epsilon_{i+1}, \tag{5.215}$$

$$\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\| \le \|\pi_\tau(z_*) - \pi_\tau(x_i)\| + \bar{q}\epsilon_{i+1}, \tag{5.216}$$

$$\|\pi_\tau(z_*) - y_t^{(i,\tau)}\| \le \|\pi_\tau(z_*) - \pi_\tau(x_i)\| + \bar{q}\epsilon_{i+1}, \tag{5.217}$$

$$\|\pi_\tau(z_*) - y_{i,\tau}\| \le \|\pi_\tau(z_*) - \pi_\tau(x_i)\| + (\bar{q}+1)\epsilon_{i+1}. \tag{5.218}$$

Using Lemmas 5.7 and 5.8 and (5.218) and arguing as in Section 5.3 we show that (5.87) holds and that

$$\|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau}))\| \le \|B_2(z_* - x_i)\| + (\bar{q}+1)\epsilon_{i+1}\text{Card}(\mathcal{E}). \tag{5.219}$$

By (5.20), (5.204), and (5.219),

$$\begin{aligned}
&\|B_2(z_* - x_{i+1})\| \\
&\le \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau}))\| + \|B_2(B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau})) - x_{i+1}\| \\
&\le \|B_2(z_* - x_i)\| + (\bar{q}+1)\epsilon_{i+1}\text{Card}(\mathcal{E}) + \epsilon_{i+1}m_0.
\end{aligned} \tag{5.220}$$

Set

$$\epsilon_0 = 0. \tag{5.221}$$

Using (5.214), (5.220), and (5.221) and arguing as in Section 5.3, we show by induction that for all integers $i \ge 0$,

$$\|B_2(z_* - x_i)\| \le 2Mm_0^{1/2} + (\sum_{j=0}^{i} \epsilon_j)((\bar{q}+1)\text{Card}(\mathcal{E}) + m_0). \tag{5.222}$$

It follows from (5.60), (5.216)–(5.218), and (5.222) that for every integer $i \ge 0$, every $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, every $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and every $j = 0, \dots, p(t)$,

$$\begin{aligned}
&\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\| \\
&\le 2Mm_0^{1/2} + (\sum_{j=0}^{i+1} \epsilon_j)((\bar{q}+1)\text{Card}(\mathcal{E}) + m_0) \\
&\le 2Mm_0^{1/2} + \Lambda((\bar{q}+1)\text{Card}(\mathcal{E}) + m_0),
\end{aligned} \tag{5.223}$$

$$\|\pi_\tau(z_*) - y_t^{(i,\tau)}\|$$
$$\le 2Mm_0^{1/2} + (\sum_{j=0}^{i+1} \epsilon_j)((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0)$$
$$\le 2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0), \tag{5.224}$$
$$\|\pi_\tau(z_*) - y_{i,\tau}\|$$
$$\le 2Mm_0^{1/2} + (\sum_{j=0}^{i+1} \epsilon_j)((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0)$$
$$\le 2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0). \tag{5.225}$$

It follows from (5.60), (5.193), and (5.222) that for every integer $i \ge 0$,

$$\|B_2(z_* - x_i)\| \le 2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0), \tag{5.226}$$
$$\|x_i\| \le 2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + M. \tag{5.227}$$

Set

$$E_0 = \{i \in \{n_0, n_0+1, \dots\} : \ \lambda_{i+1} \ge \gamma_0\}. \tag{5.228}$$

Assume that an integer i satisfies

$$i \ge n_0, \ \lambda_{i+1} \ge \gamma_0. \tag{5.229}$$

By (5.205), (5.208), (5.212), and (5.229), there exist

$$\tilde{\tau} = (\tilde{\tau}_1, \dots, \tilde{\tau}_{p(\tilde{\tau})}) \in \mathcal{E},$$
$$\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{p(t)}) \in \Omega_{i+1,\tilde{\tau}}$$

and

$$\tilde{j} \in \{0, \dots, p(\tilde{t}) - 1\}$$

such that

$$\gamma_0 \le \lambda_{i+1} = \lambda_{i,\tilde{\tau}} = \|y_{\tilde{t},\tilde{j}+1}^{(i,\tilde{\tau})} - y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}\|. \tag{5.230}$$

We show that

$$d_{\widehat{X}_{\tilde{\tau}}}(y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}, C_{\tilde{\tau},\tilde{t}_{\tilde{j}+1}}) \ge \gamma_0/4. \tag{5.231}$$

Assume the contrary. Then there exists

$$\xi \in C_{\tilde{\tau},\tilde{t}_{\tilde{j}+1}} \tag{5.232}$$

such that

$$\|y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}} - \xi\| < \gamma_0/4. \tag{5.233}$$

By (5.13), (5.197), (5.211), (5.229), (5.232), and (5.233),

$$\begin{aligned}
&\|y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}+1} - \xi\| \\
&\leq \|y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}+1} - P_{\tilde{\tau},\tilde{t}_{\tilde{j}+1}}(y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}})\| + \|P_{\tilde{\tau},\tilde{t}_{\tilde{j}+1}}(y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}}) - \xi\| \\
&\leq \epsilon_{i+1} + \|y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}} - \xi\| \\
&< \epsilon_{i+1} + \gamma_0/4 < \gamma_0/2.
\end{aligned} \tag{5.234}$$

In view of (5.233) and (5.234),

$$\begin{aligned}
&\|y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}+1} - \|y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}}\| \\
&\leq \|y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}+1} - \xi\| + \|\xi - y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}}\| < \gamma_0/4 + \gamma_0/2.
\end{aligned}$$

This contradicts (5.230). The contradiction we have reached proves that (5.231) holds.

Property (P1), (5.193), (5.223), and (5.231) imply that

$$\|\pi_{\tilde{\tau}}(z_*) - P_{\tilde{\tau},\tilde{t}_{\tilde{j}+1}}(y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}})\| \leq \|\pi_{\tilde{\tau}}(z_*) - y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}})\| - \gamma. \tag{5.235}$$

It follows from (5.197), (5.210), (5.229), and (5.235) that

$$\begin{aligned}
&\|\pi_{\tilde{\tau}}(z_*) - y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}+1})\| \\
&\leq \epsilon_{i+1} - \gamma + \|\pi_{\tilde{\tau}}(z_*) - y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}})\| \\
&\leq -3\gamma/4 + \|\pi_{\tilde{\tau}}(z_*) - y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}})\|.
\end{aligned} \tag{5.236}$$

By (5.209), (5.211), (5.215), and (5.236),

$$\begin{aligned}
&\|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - \|\pi_{\tilde{\tau}}(z_*) - y^{(i,\tilde{\tau})}_{\tilde{t}}\| \\
&= \|\pi_{\tilde{\tau}}(z_*) - y^{(i,\tilde{\tau})}_{\tilde{t},0}\| - \|\pi_{\tilde{\tau}}(z_*) - y^{(i,\tilde{\tau})}_{\tilde{t},p(\tilde{t})}\|
\end{aligned}$$

$$= \sum_{j=0}^{p(\tilde{t})-1} [\|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},j}^{(i,\tilde{\tau})}\| - \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},j+1}^{(i,\tilde{\tau})}\|]$$

$$= \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}\| - \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},\tilde{j}+1}^{(i,\tilde{\tau})}\|$$

$$+ \sum \{\|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},j}^{(i,\tilde{\tau})}\| - \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},j+1}^{(i,\tilde{\tau})}\| :$$

$$j \in \{0, \dots, p(\tilde{t})\} \setminus \{\tilde{j}\}\} \geq 3\gamma/4 - \epsilon_{i+1}(\bar{q} - 1),$$

$$\|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t}}^{(i,\tilde{\tau})}\| \leq \|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - 3\gamma/4 + \epsilon_{i+1}(\bar{q} - 1). \tag{5.237}$$

It follows from (5.28), (5.29), (5.197), (5.207), (5.217), (5.229), (5.237), and the convexity of the norm $\| \cdot \|$ that

$$\|\pi_{\tilde{\tau}}(z_*) - y_{i,\tilde{\tau}}\|$$

$$\leq \|\pi_\tau(z_*) - \sum_{t \in \Omega_{i+1,\tilde{\tau}}} w_{i+1,\tilde{\tau}}(t) y_t^{(i,\tilde{\tau})}\|$$

$$+\| \sum_{t \in \Omega_{i+1,\tilde{\tau}}} w_{i+1,\tilde{\tau}}(t) y_t^{(i,\tilde{\tau})} - y_{i,\tilde{\tau}}\|$$

$$\leq \sum_{t \in \Omega_{i+1,\tilde{\tau}}} w_{i+1,\tilde{\tau}}(t) \|\pi_{\tilde{\tau}}(z_*) - y_t^{(i,\tilde{\tau})}\| + \epsilon_{i+1}$$

$$\leq w_{i+1,\tilde{\tau}}(\tilde{t})[\|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - 3\gamma/4 + \epsilon_{i+1}(\bar{q} - 1)]$$

$$+ \sum \{w_{i+1,\tilde{\tau}}(t) \|\pi_{\tilde{\tau}}(z_*) - y_t^{(i,\tilde{\tau})}\| : \ t \in \Omega_{i+1,\tilde{\tau}} \setminus \{\tilde{t}\}\} + \epsilon_{i+1}$$

$$\leq w_{i+1,\tilde{\tau}}(\tilde{t}) \|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\|$$

$$-3\gamma\Delta/4 + w_{i+1,\tilde{\tau}}(\tilde{t})\epsilon_{i+1}(\bar{q} - 1)$$

$$+ \sum \{w_{i+1,\tilde{\tau}}(t)(\|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| + \bar{q}\epsilon_{i+1}) : t \in \Omega_{i+1,\tilde{\tau}} \setminus \{\tilde{t}\}\} + \epsilon_{i+1}$$

$$\leq \|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - 3\gamma\Delta/4 + (\bar{q} + 1)\epsilon_{i+1}$$

$$\leq \|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - 2^{-1}\gamma\Delta. \tag{5.238}$$

Lemmas 5.7 and 5.8, (5.197), (5.226), (5.229), and (5.238) imply that

$$\|B_2(z_* - B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau}))\|^2$$

$$\leq \sum_{\tau \in \mathcal{E}} \|\pi_\tau(z_*) - y_{i,\tau}\|^2$$

$$\leq \|\pi_{\tilde{\tau}}(z_*) - y_{i,\tilde{\tau}}\|^2$$

$$+\sum\{\|\pi_\tau(z_*) - y_{i,\tau}\|^2 : \ \tau \in \mathcal{E} \setminus \{\tilde{\tau}\}\}$$
$$\leq (\|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - 2^{-1}\gamma\Delta)^2$$
$$+\sum\{\|\pi_\tau(z_*) - \pi_\tau(x_i)\| + (1+\bar{q})\epsilon_{i+1}^2 : \ \tau \in \mathcal{E} \setminus \{\tilde{\tau}\}\}$$
$$\leq \|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\|^2 - 4^{-1}\gamma^2\Delta^2$$
$$+\sum\{\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2$$
$$+(1+\bar{q})^2\epsilon_{i+1}^2 + 2(1+\bar{q})\epsilon_{i+1}\|\pi_\tau(z_*) - \pi_\tau(x_i)\| : \ \tau \in \mathcal{E} \setminus \{\tilde{\tau}\}\}$$
$$\leq \sum_{\tau\in\mathcal{E}} \|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 - 4^{-1}\gamma^2\Delta^2$$
$$+2(1+\bar{q})\epsilon_{i+1}(2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + 1)\mathrm{Card}(\mathcal{E})$$
$$\leq \|B_2(z_* - x_i)\|^2 - 4^{-1}\gamma^2\Delta^2$$
$$+2(1+\bar{q})\epsilon_{i+1}(2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + 1)\mathrm{Card}(\mathcal{E}). \tag{5.239}$$

In view of (5.20), (5.36), (5.204), and (5.226),

$$|\|B_2(z_* - x_{i+1})\|^2 - \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau}))\|^2|$$
$$\leq \|B_2(x_{i+1} - B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau}))\|$$
$$\times(2\|B_2(z_* - x_{i+1})\| + \|B_2(x_{i+1} - B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau}))\|)$$
$$\leq \epsilon_{i+1}m_0^{1/2}(4Mm_0^{1/2} + 2\Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + \epsilon_{i+1}m_0^{1/2}). \tag{5.240}$$

It follows from (5.195), (5.239), and (5.240) that

$$\|B_2(z_* - x_{i+1})\|^2$$
$$\leq \|B_2(z_* - x_i)\|^2 - 4^{-1}\gamma^2\Delta^2$$
$$+\epsilon_{i+1}(2\bar{q}(2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + 1)\mathrm{Card}(\mathcal{E})$$
$$+4Mm_0 + 2\Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E})m_0^{1/2}) + m_0) + m_0)$$
$$\leq \|B_2(z_* - x_i)\|^2 - 4^{-1}\gamma^2\Delta^2$$
$$+\epsilon_{i+1}(4(\bar{q} + m_0^{1/2})Mm_0^{1/2} + 2\Lambda(\bar{q}+1)\mathrm{Card}(\mathcal{E}) + 2m_0 + 2)\mathrm{Card}(\mathcal{E})$$
$$\leq \|B_2(z_* - x_i)\|^2 - 4^{-1}\gamma^2\Delta^2 + \epsilon_{i+1}M_1 \tag{5.241}$$

for each integer i satisfying (5.229).

By (5.196), (5.220), and (5.226), for each integer $i \ge n_0$,

$$\|B_2(z_* - x_{i+1})\|^2 - \|B_2(z_* - x_i)\|^2$$
$$(\|B_2(z_* - x_{i+1})\| - \|B_2(z_* - x_i)\|)(\|B_2(z_* - x_{i+1})\| + \|B_2(z_* - x_i)\|)$$
$$\le 2((\bar{q}+1)\epsilon_{i+1}\mathrm{Card}(\mathcal{E}) + \epsilon_{i+1}m_0)(2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0)$$
$$\le 2\epsilon_{i+1}((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0)(2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0)$$
$$\le \epsilon_{i+1}M_2. \tag{5.242}$$

By (5.60), (5.226), (5.228), and (5.241), for each integer $n > n_0$,

$$(2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E})) + m_0 + M)^2$$
$$\ge \|B_2(z_* - x_{n_0})\|^2$$
$$\ge \|B_2(z_* - x_{n_0})\|^2 - \|B_2(z_* - x_n)\|^2$$
$$= \sum_{i=0}^{n-1}(\|B_2(z_* - x_i)\|^2 - \|B_2(z_* - x_{i+1})\|^2)$$
$$\ge \sum\{\|B_2(z_* - x_i)\|^2 - \|B_2(z_* - x_{i+1})\|^2 :\ i \in E_0 \cap [n_0, n-1]\}$$
$$+ \sum\{\|B_2(z_* - x_i)\|^2 - \|B_2(z_* - x_{i+1})\|^2 :\ i \in \{n_0, \dots, n-1\} \setminus E_0\}$$
$$\ge 4^{-1}\gamma^2\Delta^2\mathrm{Card}(E_0 \cap [n_0, n-1]) - M_1\sum\{\epsilon_{i+1} :\ i \in E_0 \cap [n_0, n-1]\}$$
$$-M_2\sum\{\epsilon_{i+1} :\ i \in \{n_0, \dots, n-1\} \setminus E_0\}$$

and

$$4^{-1}\gamma^2\Delta^2\mathrm{Card}(E_0 \cap [n_0, n-1])$$
$$\le \Lambda(M_1 + M_2) + (2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + M)^2.$$

Since the relation above holds for any natural number $n > n_0$ we conclude that

$$\mathrm{Card}(E_0)$$
$$\le 4\gamma^{-2}\Delta^{-2}(\Lambda(M_1 + M_2) + (2Mm_0^{1/2} + \Lambda((\bar{q}+1)\mathrm{Card}(\mathcal{E}) + m_0) + M)^2). \tag{5.243}$$

Assume that an integer $i \ge 0$ satisfies

$$i \ge n_0,\ \lambda_{i+1} < \gamma_0. \tag{5.244}$$

By (5.12), (5.15), (5.197), (5.205), (5.208), (5.211), (5.212), and (5.244), for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$, and each $j = 0, \dots, p(t) - 1$,

$$\gamma_0 > \lambda_{i+1} \ge \lambda_{i,\tau} \ge \|y_{t,j+1}^{(i,\tau)} - y_{t,j}^{(i,\tau)}\|, \tag{5.245}$$

$$\begin{aligned} &\|y_{t,j}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\| \\ &\le \|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\| + \|y_{t,j+1}^{(i,\tau)} - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\| \\ &< \gamma_0 + \epsilon_{i+1} < 2\gamma_0 \end{aligned} \tag{5.246}$$

and

$$d_{\widehat{X}_\tau}(y_{t,j}^{(i,\tau)}, C_{\tau,t_{j+1}}) < 2\gamma_0. \tag{5.247}$$

Let

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}.$$

In view of (5.209) and (5.245), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j = 0, 1, \dots, p(t)$,

$$\|\pi_\tau(x_i) - y_{t,j}^{(i,\tau)}\| \le \gamma_0 j \le \gamma_0 \bar{q}. \tag{5.248}$$

By (5.247) and (5.248), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j = 0, 1, \dots, p(t) - 1$,

$$\begin{aligned} d_{\widehat{X}_\tau}(\pi_\tau(x_i), C_{\tau,t_{j+1}}) &\le \|\pi_\tau(x_i) - y_{t,j}^{(i,\tau)}\| + d_{\widehat{X}_\tau}(y_{t,j}^{(i,\tau)}, C_{\tau,t_{j+1}}) \\ &< \gamma_0(\bar{q} + 2). \end{aligned} \tag{5.249}$$

By (5.7), (5.8), and (5.249), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j = 1, \dots, p(t)$,

$$d_X(x_i, C_{t_j}) \le \gamma_0(\bar{q} + 2).$$

Together with (5.3), (5.25), (5.26), and (5.194) this implies that

$$d_X(x_i, C_s) \le \gamma_0(\bar{q} + 2) = \epsilon, \ \ s = 1, \dots, m \tag{5.250}$$

for all integers i satisfying (5.244).

By (5.198), (5.228), (5.243), (5.244), and (5.250),

$$\text{Card}(\{i \in \{0, 1, \dots\} : \max\{d_X(x_i, C_s) : s = 1, \dots, m\} > \epsilon\})$$
$$\leq n_0 + \text{Card}(\{i \in \{n_0, n_0 + 1, \dots\} : \lambda_{i+1} \geq \gamma_0\})$$
$$\leq n_0 + 4\Delta^{-2}\gamma^{-2}(\Lambda(M_1 + m_2) + (2Mm_0^{1/2} + \Lambda((\bar{q} + 1)\text{Card}(\mathcal{E}) + m_0 + M)^2) = Q.$$

Theorem 5.4 is proved.

5.8 Proof of Theorem 5.5

In view of (5.62), there exists

$$z_* = (z_{*,1}, \dots, z_{*,l}) \in B_X(0, M) \cap C. \tag{5.251}$$

Set

$$\gamma_1 = \epsilon(\bar{q} + 1)^{-1}. \tag{5.252}$$

By (A2) there exists $\gamma_0 \in (0, \gamma_1)$ such that the following property holds:

(P2) for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$, each $z \in C_{\tau,s}$ satisfying $\|z\| \leq M$, and each $x \in \widehat{X}_\tau$ satisfying

$$\|x\| \leq M + 2Mm_0^{1/2}$$

and

$$d_{\widehat{X}_\tau}(x, C_{\tau,s}) \geq \gamma_1/4$$

the inequality

$$\|z - P_{\tau,s}(x)\| \leq \|z - x\| - \phi(\gamma_0)$$

holds where the function ϕ is defined by

$$\phi(r) = 2M^{1/2}m_0^{1/2}\text{Card}(\mathcal{E})^{1/2}\bar{N}\bar{q}\Delta^{-1}r^{1/2} \tag{5.253}$$

for all $r \geq 0$.

By (A2) there exists $\gamma \in (0, \gamma_0)$ such that the following property holds:

(P3) for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$, each $z \in C_{\tau,s}$ satisfying $\|z\| \leq M$, and each $x \in \widehat{X}_\tau$ satisfying

$$\|x\| \leq M + 2Mm_0^{1/2}$$

and

$$d_{\widehat{X}_\tau}(x, C_{\tau,s}) \geq \gamma_0/4$$

the inequality

$$\|z - P_{\tau,s}(x)\| \leq \|z - x\| - \gamma$$

holds.

Set

$$Q = \bar{N}(4M^2 m_0 \Delta^{-2} \gamma^{-2} + 1). \tag{5.254}$$

Assume that for all natural numbers i,

$$(\Omega_{i,\tau}, w_{i,\tau}) \in \mathcal{M}_\tau, \ \ \tau \in \mathcal{E}, \tag{5.255}$$

$$(\Omega_{i,\tau}, w_{i,\tau}) = (\Omega_{i+\bar{N},\tau}, w_{i+\bar{N},\tau}), \ \ \tau \in \mathcal{E}, \tag{5.256}$$

$$x_0 \in B_X(0, M) \tag{5.257}$$

and that sequences $\{x_i\}_{i=1}^\infty \subset X$, $\{\lambda_i\}_{i=1}^\infty \subset [0, \infty)$ satisfy for each integer $i \geq 1$,

$$(x_i, \lambda_i) \in A(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau \in \mathcal{E}}, 0). \tag{5.258}$$

It follows from (5.22), (5.28), (5.30), (5.61), and (5.255) that for every integer $i \geq 0$, every $\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \mathcal{E}$, every $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1,\tau}$,

$$\|P[t](x) - P[t](y)\| \leq \|x - y\| \tag{5.259}$$

for all $x, y \in \widehat{X}_\tau$ and

$$\|P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(x) - P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(y)\| \leq \|x - y\| \tag{5.260}$$

for all $x, y \in \widehat{X}_\tau$.

Let $i \geq 0$ be an integer. By (5.258),

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, \{(\Omega_{i+1,\tau}, w_{i+1,\tau})\}_{\tau \in \mathcal{E}}, 0). \tag{5.261}$$

In view of (5.35) and (5.261), there exist

$$(y_{i,\tau}, \lambda_{i,\tau}) \in A_\tau(\pi_\tau(x_i), (\Omega_{i+1,\tau}, w_{i+1,\tau}), 0), \ \ \tau \in \mathcal{E} \tag{5.262}$$

such that

$$x_{i+1} = B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau}), \ \ \lambda_{i+1} = \max\{\lambda_{i,\tau} : \ \tau \in \mathcal{E}\}. \tag{5.263}$$

By (5.34) and (5.262), for each $\tau \in \mathcal{E}$, there exist

$$(y_t^{(i,\tau)}, \lambda_t^{(i,\tau)}) \in A_{\tau,0}(\pi_\tau(x_i), t, 0),\ \ t \in \Omega_{i+1,\tau} \tag{5.264}$$

such that

$$y_{i,\tau} = \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}, \tag{5.265}$$

$$\lambda_{i,\tau} = \max\{\lambda_t^{(i,\tau)} :\ t \in \Omega_{i+1,\tau}\}. \tag{5.266}$$

By (5.33) and (5.264), for each $\tau \in \mathcal{E}$ and each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ there exists a sequence

$$\{y_{t,j}^{(i,\tau)}\}_{j=0}^{p(t)} \subset \widehat{X}_\tau$$

such that

$$y_{t,0}^{(i,\tau)} = \pi_\tau(x_i), \tag{5.267}$$

for all $j = 1, \dots, p(t)$,

$$y_{t,j}^{(i,\tau)} = P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)}), \tag{5.268}$$

$$y_t^{(i,\tau)} = y_{t,p(t)}^{(i,\tau)}, \tag{5.269}$$

$$\lambda_t^{(i,\tau)} = \max\{\|y_{t,j}^{(i,\tau)} - y_{t,j-1}^{(i,\tau)}\| :\ j = 1, \dots, p(t)\}. \tag{5.270}$$

It follows from (5.30), (5.263), (5.265), and (5.267)–(5.269) that for each $\tau \in \mathcal{E}$ and each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$,

$$\begin{aligned} y_t^{(i,\tau)} &= P_{\tau,t_{p(t)}} \cdots P_{\tau,t_{(1)}}(y_{t,0}^{(i,\tau)}) \\ &= P_{\tau,t_{p(t)}} \cdots P_{\tau,t_{(1)}}(\pi_\tau(x_i)) = P[t](\pi_\tau(x_i)), \end{aligned} \tag{5.271}$$

$$y_{i,\tau} = \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) P[t](\pi_\tau(x_i)) = P_{\Omega_{i+1},w_{i+1}}(\pi_\tau(x_i)), \tag{5.272}$$

$$x_{i+1} = B_1(\sum_{\tau \in \mathcal{E}} P_{\Omega_{i+1,\tau},w_{i+1,\tau}}(\pi_\tau(x_i))), \tag{5.273}$$

$$B_2(x_{i+1}) = B_2 \circ B_1(\sum_{\tau \in \mathcal{E}} P_{\Omega_{i+1,\tau},w_{i+1,\tau}}(\pi_\tau(x_i))). \tag{5.274}$$

By (5.12), (5.20), (5.30), and (5.251), for each $\tau \in \mathcal{E}$ and each

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau},$$

we have

$$P[t](\pi_\tau(z_*)) = \pi_\tau(z_*), \tag{5.275}$$

$$P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(\pi_\tau(z_*)) = \pi_\tau(z_*) \tag{5.276}$$

and

$$B_1(\sum_{\tau \in \mathcal{E}} P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(\pi_\tau(z_*))) = B_1(\sum_{\tau \in \mathcal{E}} \pi_\tau(z_*)) = z_*. \tag{5.277}$$

Relations (5.125), (5.274), and (5.277) imply that

$$\|B_2(z_* - x_{i+1})\| \le \|B_2(z_* - x_i)\|. \tag{5.278}$$

Let

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}.$$

By (5.13), (5.251), and (5.268), for each integer j satisfying $0 \le j < p(t)$,

$$\begin{aligned} &\|\pi_\tau(z_*) - y^{(i,\tau)}_{t,j+1}\| \\ &= \|\pi_\tau(z_*) - P_{\tau,t_{j+1}}(y^{(i,\tau)}_{t,j})\| \\ &\le \|\pi_\tau(z_*) - y^{(i,\tau)}_{t,j}\|. \end{aligned} \tag{5.279}$$

In view of (5.267), (5.269), and (5.279), for all integers $j \in \{0, \dots, p(t)\}$,

$$\begin{aligned} \|\pi_\tau(z_*) - y^{(i,\tau)}_{t,j}\| &\le \|\pi_\tau(z_*) - y^{(i,\tau)}_{t,0}\| \\ &= \|\pi_\tau(z_*) - \pi_\tau(x_i)\| \end{aligned} \tag{5.280}$$

and

$$\|\pi_\tau(z_*) - y^{(i,\tau)}_t\| \le \|\pi_\tau(z_*) - \pi_\tau(x_i)\|. \tag{5.281}$$

By (5.265), (5.281), and the convexity of the norm,

$$\|\pi_\tau(z_*) - y_{i,\tau}\| \le \|\pi_\tau(z_*) - \pi_\tau(x_i)\|. \tag{5.282}$$

In view of (5.20), (5.251), (5.257), and (5.278),

$$\|x_i - z_*\| \le \|B_2(z_* - x_0)\| \le 2m_0^{1/2}M, \tag{5.283}$$

$$\|x_i\| \le 2m_0^{1/2}M + M. \tag{5.284}$$

By (5.280)–(5.283), for each integer $i \ge 0$, for each $\tau \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$, and each $j = 0, \dots, p(t)$,

$$\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\| \le 2m_0^{1/2}M, \tag{5.285}$$

$$\|\pi_\tau(z_*) - y_t^{(i,\tau)}\| \le 2Mm_0^{1/2}, \tag{5.286}$$

$$\|\pi_\tau(z_*) - y_{i,\tau}\| \le 2Mm_0^{1/2}. \tag{5.287}$$

For each integer $i \ge 0$ and each $x \in X$, set

$$T_{i+1}(x) = B_1(\sum_{\tau \in \mathcal{E}} P_{\Omega_{i+1,\tau}, w_{i+1,\tau}}(\pi_\tau(x))). \tag{5.288}$$

In view of (5.256) and (5.288), for each integer $i \ge 1$,

$$T_{i+\bar{N}} = T_i. \tag{5.289}$$

Relations (5.125), (5.273), and (5.288) imply that for each integer $i \ge 0$,

$$x_{i+1} = T_{i+1}(x_i) \tag{5.290}$$

and that for all $x, y \in X$,

$$\|B_2(T_i(x)) - B_2(T_i(y))\| \le \|B_2(x - y)\|. \tag{5.291}$$

Set

$$E_0 = \{i \in \{0, 1, \dots\} : \lambda_{i+1} \ge \gamma_0\}. \tag{5.292}$$

Assume that an integer $i \ge 0$ satisfies

$$\lambda_{i+1} \ge \gamma_0. \tag{5.293}$$

By (5.263), (5.266), (5.270), and (5.293), there exist

$$\tilde{\tau} = (\tilde{\tau}_1, \dots, \tilde{\tau}_{p(\tilde{\tau})}) \in \mathcal{E},$$

$$\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{p(\tilde{t})}) \in \Omega_{i+1,\tilde{\tau}}$$

and

$$\tilde{j} \in \{0, \dots, p(\tilde{t}) - 1\}$$

such that

$$\gamma_0 \le \lambda_{i+1} = \lambda_{i,\tilde{\tau}} = \lambda_{\tilde{t}}^{(i,\tilde{\tau})} = \|y_{\tilde{t},\tilde{j}+1}^{(i,\tilde{\tau})} - y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}\|. \tag{5.294}$$

We show that

$$d_{\widehat{X}_{\tilde{\tau}}}(y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}, C_{\tilde{\tau},\tilde{t}_{\tilde{j}+1}}) \ge \gamma_0/4. \tag{5.295}$$

Assume the contrary. Then there exists

$$\xi \in C_{\tilde{\tau},\tilde{t}_{\tilde{j}+1}} \tag{5.296}$$

such that

$$\|y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})} - \xi\| < \gamma_0/4. \tag{5.297}$$

By (5.13), (5.268), and (5.296),

$$\begin{aligned} &\|y_{\tilde{t},\tilde{j}+1}^{(i,\tilde{\tau})} - \xi\| \\ &= \|P_{\tilde{\tau},\tilde{t}_{\tilde{j}+1}}(y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}) - \xi\| \\ &\le \|y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})} - \xi\| < \gamma_0/4. \end{aligned} \tag{5.298}$$

In view of (5.297) and (5.298),

$$\begin{aligned} &\|y_{\tilde{t},\tilde{j}+1}^{(i,\tilde{\tau})} - \|y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}\| \\ &\le \|y_{\tilde{t},\tilde{j}+1}^{(i,\tilde{\tau})} - \xi\| + \|\xi - y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}\| < \gamma_0/2. \end{aligned}$$

This contradicts (5.294). The contradiction we have reached proves that (5.295) holds.

Property (P3), (5.251), (5.268), and (5.285) imply that

$$\begin{aligned} &\|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},\tilde{j}+1}^{(i,\tilde{\tau})}\| \\ &= \|\pi_{\tilde{\tau}}(z_*) - P_{\tilde{\tau},\tilde{t}_{\tilde{j}+1}}(y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})})\| \\ &\le \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}\| - \gamma. \end{aligned} \tag{5.299}$$

By (5.267), (5.269), (5.279), and (5.299),

$$
\begin{aligned}
&\|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t}}^{(i,\tilde{\tau})}\| \\
&= \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},0}^{(i,\tilde{\tau})}\| - \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},p(\tilde{t})}^{(i,\tilde{\tau})}\| \\
&= \sum_{j=0}^{p(\tilde{t})-1} [\|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}\| - \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},\tilde{j}+1}^{(i,\tilde{\tau})}\|] \\
&\geq \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}\| - \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},\tilde{j}+1}^{(i,\tilde{\tau})}\| \geq \gamma. \qquad (5.300)
\end{aligned}
$$

It follows from (5.28), (5.29), (5.265), (5.281), (5.300), and the convexity of the norm $\|\cdot\|$ that

$$
\begin{aligned}
&\|\pi_{\tilde{\tau}}(z_*) - y_{i,\tilde{\tau}}\| \\
&= \|\pi_{\tilde{\tau}}(z_*) - \sum_{t\in\Omega_{i+1,\tilde{\tau}}} w_{i+1,\tilde{\tau}}(t) y_t^{(i,\tilde{\tau})}\| \\
&\leq \sum_{t\in\Omega_{i+1,\tilde{\tau}}} w_{i+1,\tilde{\tau}}(t)\|\pi_{\tilde{\tau}}(z_*) - y_t^{(i,\tilde{\tau})}\| \\
&\leq w_{i+1,\tilde{\tau}}(\tilde{t})[\|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - \gamma] \\
&+ \sum\{w_{i+1,\tilde{\tau}}(t)\|\pi_{\tilde{\tau}}(z_*) - y_t^{(i,\tilde{\tau})}\| : \ t \in \Omega_{i+1,\tilde{\tau}} \setminus \{\tilde{t}\}\} \\
&\leq \|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - \gamma\Delta. \qquad (5.301)
\end{aligned}
$$

Lemmas 5.7 and 5.8, (5.263), and (5.301) imply that

$$
\begin{aligned}
&\|B_2(z_*) - B_2(x_{i+1})\|^2 \\
&= \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau}))\|^2 \\
&\leq \sum_{\tau\in\mathcal{E}} \|\pi_\tau(z_*) - y_{i,\tau}\|^2 \\
&= \|\pi_{\tilde{\tau}}(z_*) - y_{i,\tilde{\tau}}\|^2 \\
&+ \sum\{\|\pi_\tau(z_*) - y_{i,\tau}\|^2 : \ \tau \in \mathcal{E} \setminus \{\tilde{\tau}\}\} \\
&\leq (\|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - \gamma\Delta)^2 \\
&+ \sum\{\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 : \ \tau \in \mathcal{E} \setminus \{\tilde{\tau}\}\} \\
&\leq \|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\|^2 - \gamma^2\Delta^2
\end{aligned}
$$

$$+\sum\{\|\pi_\tau(z_*)-\pi_\tau(x_i)\|^2:\ \tau\in\mathcal{E}\setminus\{\tilde{\tau}\}\}$$
$$=\|B_2(z_*-x_i)\|^2-\gamma^2\Delta^2. \tag{5.302}$$

Thus we have shown that the following property holds:

(P4) for each integer $i\geq 0$ satisfying $\lambda_{i+1}\geq\gamma_0$,

$$\|B_2(z_*-x_{i+1})\|^2\leq\|B_2(z_*-x_i)\|^2-(\Delta\gamma)^2.$$

Let n be a natural number. Property (P4), (5.278), and (5.283) imply that

$$4M^2m_0\geq\|B_2(z_*-x_0)\|^2$$
$$\geq\|B_2(z_*-x_0)\|^2-\|B_2(z_*-x_{\bar{N}n})\|^2$$
$$=\sum_{k=0}^{n-1}(\|B_2(z_*-x_{k\bar{N}})\|^2-\|B_2(z_*-x_{(k+1)\bar{N}})\|^2)$$
$$=\sum_{k=0}^{n-1}(\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1}(\|B_2(z_*-x_j)\|^2-\|B_2(z_*-x_{j+1})\|^2))$$
$$\geq(\Delta\gamma)^2\mathrm{Card}(\{k\in\{0,\dots,n-1\}:$$
$$\max\{\lambda_{i+1}:\ i=k\bar{N},\dots,(k+1)\bar{N}-1\}\geq\gamma_0\})$$

and

$$\mathrm{Card}(\{k\in\{0,\dots,n-1\}:$$
$$\max\{\lambda_{i+1}:\ i=k\bar{N},\dots,(k+1)\bar{N}-1\}\geq\gamma_0\})$$
$$\leq 4M^2m_0(\Delta\gamma)^{-2}.$$

Since n is any natural number the relation above implies that

$$\mathrm{Card}(\{k\in\{0,1,\dots\}:$$
$$\max\{\lambda_{i+1}:\ i=k\bar{N},\dots,(k+1)\bar{N}-1\}\geq\gamma_0\})$$
$$\leq 4M^2m_0(\Delta\gamma)^{-2}. \tag{5.303}$$

In view of (5.303), there exists an integer $q_0\geq 0$ such that

$$q_0\leq 4M^2m_0\Delta^{-2}\gamma^{-2}+1,$$
$$\lambda_{i+1}<\gamma_0,\ i=q_0\bar{N},\dots,(q_0+1)\bar{N}-1. \tag{5.304}$$

By (5.27), (5.263), (5.266), (5.269), (5.270), (5.287), and (5.304), for each integer $i \in \{q_0\bar{N}, \dots, (q_0+1)\bar{N}-1\}$, for each $\tau \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$, and each $j = 0, \dots, p(t)-1$,

$$\gamma_0 > \lambda_{i+1} \geq \lambda_{i,\tau} \geq \lambda_t^{(i,\tau)} \geq \|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|, \tag{5.305}$$

$$\|\pi_\tau(x_i) - y_{t,j+1}^{(i,\tau)}\| \leq (j+1)\gamma_0, \tag{5.306}$$

$$\|\pi_\tau(x_i) - y_t^{(i,\tau)}\| \leq p(t)\gamma_0 \leq \bar{q}\gamma_0. \tag{5.307}$$

By (5.28), (5.265), (5.307), and the convexity of the norm, for each $i \in \{q_0\bar{N}, \dots, (q_0+1)\bar{N}-1\}$ and each $\tau \in \mathcal{E}$,

$$\begin{aligned} &\|\pi_\tau(x_i) - y_{i,\tau}\| \\ &= \|\pi_\tau(x_i) - \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\| \\ &\leq \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t)\|\pi_\tau(x_i) - y_t^{(i,\tau)}\| \leq \bar{q}\gamma_0. \end{aligned} \tag{5.308}$$

Lemma 5.8, (5.263), and (5.308) imply that for each $i \in \{q_0\bar{N}, \dots, (q_0+1)\bar{N}-1\}$,

$$\begin{aligned} &\|B_2(x_i - x_{i+1})\|^2 \\ &= \|B_2(x_i - B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau}))\|^2 \\ &\leq \sum_{\tau \in \mathcal{E}} \|\pi_\tau(x_i) - y_{i,\tau}\|^2 \leq \mathrm{Card}(\mathcal{E})\bar{q}^2\gamma_0^2, \end{aligned} \tag{5.309}$$

$$\|x_i - x_{i+1}\| \leq \mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0 \tag{5.310}$$

$$\|x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}}\| \leq \bar{N}\mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0. \tag{5.311}$$

By (5.125), (5.289), and (5.291), for each integer $q > q_0$,

$$\begin{aligned} &\|B_2(x_{q\bar{N}} - x_{(q+1)\bar{N}})\| \\ &= \|B_2(\prod_{j=q_0\bar{N}+1}^{q\bar{N}} T_j(x_{q_0\bar{N}}) - \prod_{j=(q_0+1)\bar{N}+1}^{(q+1)\bar{N}} T_j(x_{(q_0+1)\bar{N}})\| \\ &= \|B_2(\prod_{j=q_0\bar{N}+1}^{q\bar{N}} T_j(x_{q_0\bar{N}}) - \prod_{j=q_0\bar{N}+1}^{q\bar{N}} T_j(x_{(q_0+1)\bar{N}}))\| \end{aligned}$$

$$\leq \|B_2(x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}})\|. \tag{5.312}$$

In view of (5.20), (5.311), and (5.312), for all integers $q \geq q_0$.

$$\|B_2(x_{q\bar{N}} - x_{(q+1)\bar{N}})\| \leq \|B_2(x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}})\|$$
$$\leq m_0^{1/2}\bar{N}\mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0. \tag{5.313}$$

Let $q \geq q_0$ be an integer. It follows from (5.313) that

$$m_0^{1/2}\bar{N}\mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0$$
$$\geq \|B_2(x_{q\bar{N}} - x_{(q+1)\bar{N}})\|$$
$$\geq \|B_2(z_* - x_{q\bar{N}})\| - \|B_2(z_* - x_{(q+1)\bar{N}})\|$$
$$= \sum_{i=q\bar{N}}^{(q+1)\bar{N}-1} [\|B_2(z_* - x_i)\| - \|B_2(z_* - x_{i+1})\|]$$
$$\geq \max\{\|B_2(z_* - x_i)\| - \|B_2(z_* - x_{i+1})\| : \ i = q\bar{N}, \ldots, (q+1)\bar{N} - 1\}. \tag{5.314}$$

Let $i \in \{q\bar{N}, \ldots, (q+1)\bar{N} - 1\}$. In view of (5.314),

$$\|B_2(z_* - x_i)\| - \|B_2(z_* - x_{i+1})\| \leq m_0^{1/2}\bar{N}\mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0. \tag{5.315}$$

We show that

$$\lambda_{i+1} \leq \gamma_1.$$

Assume the contrary. Then

$$\lambda_{i+1} > \gamma_1. \tag{5.316}$$

By (5.263), (5.266), (5.266), (5.270), and (5.316), there exist

$$\tilde{\tau} = (\tilde{\tau}_1, \ldots, \tilde{\tau}_{p(\tilde{\tau})}) \in \mathcal{E},$$
$$\tilde{t} = (\tilde{t}_1, \ldots, \tilde{t}_{p(\tilde{t})}) \in \Omega_{i+1,\tilde{\tau}}$$

and

$$\tilde{j} \in \{0, \ldots, p(\tilde{t}) - 1\}$$

such that

$$\gamma_1 < \lambda_{i+1} = \lambda_{i,\tilde{\tau}} = \lambda_{\tilde{t}}^{(i,\tilde{\tau})} = \|y_{\tilde{t},\tilde{j}+1}^{(i,\tilde{\tau})} - y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}\|. \tag{5.317}$$

We show that

$$d_{\widehat{X}_{\tilde{\tau}}}(y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}}, C_{\tilde{\tau},\tilde{t}_{\tilde{j}+1}}) \geq \gamma_1/4. \tag{5.318}$$

Assume the contrary. Then there exists

$$\xi \in C_{\tilde{\tau},\tilde{t}_{\tilde{j}+1}} \tag{5.319}$$

such that

$$\|y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}} - \xi\| < \gamma_1/4. \tag{5.320}$$

By (5.13), (5.269), (5.319), and (5.320),

$$\begin{aligned}
&\|y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}+1} - \xi\| \\
&= \|P_{\tilde{\tau},t_{\tilde{j}+1}}(y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}}) - \xi\| \\
&\leq \|y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}} - \xi\| < \gamma_1/4.
\end{aligned} \tag{5.321}$$

In view of (5.320) and (5.321),

$$\begin{aligned}
&\|y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}+1} - y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}}\| \\
&\leq \|y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}+1} - \xi\| + \|\xi - y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}}\| < \gamma_1/2.
\end{aligned}$$

This contradicts (5.317). The contradiction we have reached proves that (5.318) holds.

Property (P2), (5.251), (5.268), (5.285), and (5.318) imply that

$$\begin{aligned}
&\|\pi_{\tilde{\tau}}(z_*) - y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}+1}\| \\
&= \|\pi_{\tilde{\tau}}(z_*) - P_{\tilde{\tau},\tilde{t}_{\tilde{j}+1}}(y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}})\| \\
&\leq \|\pi_{\tilde{\tau}}(z_*) - y^{(i,\tilde{\tau})}_{\tilde{t},\tilde{j}}\| - \phi(\gamma_0).
\end{aligned} \tag{5.322}$$

By (5.267), (5.269), (5.279), and (5.322),

$$\begin{aligned}
&\|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - \|\pi_{\tilde{\tau}}(z_*) - y^{(i,\tilde{\tau})}_{\tilde{t}}\| \\
&= \|\pi_{\tilde{\tau}}(z_*) - y^{(i,\tilde{\tau})}_{\tilde{t},0}\| - \|\pi_{\tilde{\tau}}(z_*) - y^{(i,\tilde{\tau})}_{\tilde{t},p(\tilde{t})}\|
\end{aligned}$$

$$= \sum_{j=0}^{p(\tilde{t})-1} [\|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}\| - \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},\tilde{j}+1}^{(i,\tilde{\tau})}\|]$$

$$\geq \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},\tilde{j}}^{(i,\tilde{\tau})}\| - \|\pi_{\tilde{\tau}}(z_*) - y_{\tilde{t},\tilde{j}+1}^{(i,\tilde{\tau})}\| \geq \phi(\gamma_0). \quad (5.323)$$

It follows from (5.28), (5.29), (5.265), (5.281), (5.323), and the convexity of the norm $\|\cdot\|$ that

$$\|\pi_{\tilde{\tau}}(z_*) - y_{i,\tilde{\tau}}\|$$

$$= \|\pi_{\tilde{\tau}}(z_*) - \sum_{t \in \Omega_{i+1,\tilde{\tau}}} w_{i+1,\tilde{\tau}}(t) y_t^{(i,\tilde{\tau})}\|$$

$$\leq \sum_{t \in \Omega_{i+1,\tilde{\tau}}} w_{i+1,\tilde{\tau}}(t) \|\pi_{\tilde{\tau}}(z_*) - y_t^{(i,\tilde{\tau})}\|$$

$$\leq w_{i+1,\tilde{\tau}}(\tilde{t})[\|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - \phi(\gamma_0)]$$

$$+ \sum \{w_{i+1,\tilde{\tau}}(t) \|\pi_{\tilde{\tau}}(z_*) - y_t^{(i,\tilde{\tau})}\| : \ t \in \Omega_{i+1,\tilde{\tau}} \setminus \{\tilde{t}\}\}$$

$$\leq \|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - \phi(\gamma_0)\Delta. \quad (5.324)$$

Lemmas 5.7 and 5.8, (5.181), (5.263), and (5.324) imply that

$$\|B_2(z_* - x_{i+1})\|^2$$

$$\leq \|B_2(z_* - B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau}))\|^2$$

$$\leq \sum_{\tau \in \mathcal{E}} \|\pi_\tau(z_*) - y_{i,\tau}\|^2$$

$$= \|\pi_{\tilde{\tau}}(z_*) - y_{i,\tilde{\tau}}\|^2$$

$$+ \sum \{\|\pi_\tau(z_*) - y_{i,\tau}\|^2 : \ \tau \in \mathcal{E} \setminus \{\tilde{\tau}\}\}$$

$$\leq (\|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\| - \phi(\gamma_0)\Delta)^2$$

$$+ \sum \{\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 : \ \tau \in \mathcal{E} \setminus \{\tilde{\tau}\}\}$$

$$\leq \|\pi_{\tilde{\tau}}(z_*) - \pi_{\tilde{\tau}}(x_i)\|^2 - \phi(\gamma_0)^2 \Delta^2$$

$$+ \sum \{\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 : \ \tau \in \mathcal{E} \setminus \{\tilde{\tau}\}\}$$

$$= \|B_2(z_* - x_i)\|^2 - \phi(\gamma_0)^2 \Delta^2. \quad (5.325)$$

By (5.278), (5.315), and (5.325),

$$\phi(\gamma_0)^2\Delta^2 \le \|B_2(z_* - x_i)\|^2 - \|B_2(z_* - x_{i+1})\|^2$$
$$\le (\|B_2(z_* - x_i)\| - \|B_2(z_* - x_{i+1})\|)(\|B_2(z_* - x_i)\| + \|B_2(z_* - x_{i+1})\|)$$
$$\le 4Mm_0\bar{N}\mathrm{Card}(\mathcal{E})^{1/2}\bar{q}\gamma_0$$

and

$$\phi(\gamma_0) \le 2\Delta^{-1}M^{1/2}m_0^{1/2}\mathrm{Card}(\mathcal{E})^{1/4}(\bar{N}q_0\gamma_0)^{1/2}.$$

This contradicts (5.235). The contradiction we have reached proves that

$$\lambda_{i+1} \le \gamma_1. \tag{5.326}$$

By (5.15), (5.263), (5.266), (5.268), (5.270), and (5.326), for each

$$\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \mathcal{E},$$

each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j = 0, \ldots, p(t) - 1$,

$$\gamma_1 \ge \lambda_{i+1} \ge \lambda_{i,\tau} \ge \lambda_t^{(i,\tau)} \ge \|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|$$
$$= \|P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)}) - y_{t,j}^{(i,\tau)}\| \tag{5.327}$$

and

$$d_{\widehat{X}_\tau}(y_{t,j}^{(i,\tau)}, C_{\tau,t_{j+1}}) \le \gamma_1. \tag{5.328}$$

In view of (5.27), (5.267), (5.327), and (5.328),

$$\|\pi_\tau(x_i) - y_{t,j+1}^{(i,\tau)}\| \le (j+1)\gamma_1 \le \bar{q}\gamma_1 \tag{5.329}$$

and

$$d_{\widehat{X}_\tau}(\pi_\tau(x_i), C_{\tau,t_{j+1}}) \le \|\pi_\tau(x_i) - y_{t,j}^{(i,\tau)}\| + d_{\widehat{X}_\tau}(y_{t,j}^{(i,\tau)}, C_{\tau,t_{j+1}}) \le (\bar{q}+1)\gamma_1. \tag{5.330}$$

By (5.7), (5.8), and (5.330), for each integer $i \in \{q\bar{N}, \ldots, (q+1)\bar{N} - 1\}$, each $\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \mathcal{E}$, each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1,\tau}$, and each $j = 1, \ldots, p(t)$,

$$d_X(x_i, C_{t_j}) = d_{\widehat{X}_\tau}(\pi_\tau(x_i), C_{\tau,t_j}) \le (\bar{q}+1)\gamma_1.$$

Together with (5.3), (5.25), and (5.26) this implies that

$$d_X(x_i, C_s) \le (\bar{q}+1)\gamma_1$$

for all $s = 1, \dots, m$. By the relation above and (5.252), for all integers $i \geq q_0\bar{N}$ and all $s = 1, \dots, m$,

$$d_X(x_i, C_s) \leq (\bar{q} + 1)\gamma_1 = \epsilon.$$

Theorem 5.5 is proved.

5.9 Proof of Theorem 5.6

Theorem 5.6 is deduced from Theorems 2.9 and 5.5. Let $Y = X$, $N = \bar{N}$, $\rho(x, y) = \|B_2(x - y)\|$, $x, y \in X$, $\mathfrak{A}$ be the set of all mappings S defined on the set of natural numbers into the set of operators

$$B_1 \circ (\sum_{\tau \in \mathcal{E}} P_{\Omega_\tau, w_\tau} \circ \pi_\tau) : X \to X,$$

with

$$(\Omega_\tau, w_\tau) \in \mathcal{M}_\tau, \ \ \tau \in \mathcal{E}$$

such that for each integer $i \geq 1$,

$$S(i) = B_1 \circ (\sum_{\tau \in \mathcal{E}} P_{\Omega_{\tau(i)}, w_{\tau(i)}} \circ \pi_\tau),$$

where

$$(\Omega_{\tau(i)}, w_{\tau(i)}) \in \mathcal{M}_\tau, \ \ \tau \in \mathcal{E}$$

satisfy

$$(\Omega_{\tau(i+\bar{N})}, w_{\tau(i+\bar{N})}) = (\Omega_{\tau(i)}, w_{\tau(i)})$$

for all integers $i \geq 1$. Let

$$F = \{x \in X : \ d_X(x, C_s) \leq \epsilon_0/4, \ \ s = 1, \dots, m\}.$$

Theorem 5.5 implies that for every $M > 0$ there exists $Q > 0$ such that property (P6) holds. Let $Q \geq 1$ be such that property (P6) holds with $M = M_0$. Set

$$\delta = 4^{-1}(\bar{q} + 1)^{-1} m_0^{-1/2} \mathrm{Card}(\mathcal{E})^{-1} \epsilon_0 Q^{-1} (2\bar{N} + 1)^{-1}.$$

Assume that for all natural numbers i,

$$(\Omega_{i,\tau}, w_{i,\tau}) \in \mathcal{M}_\tau,\ \tau \in \mathcal{E},$$

$$(\Omega_{i,\tau}, w_{i,\tau}) = (\Omega_{i+\bar{N},\tau}, w_{i+\bar{N},\tau}),\ \tau \in \mathcal{E},$$

$$x_0 \in B_X(0, M_0)$$

and that sequences $\{x_i\}_{i=1}^\infty \subset X$, $\{\lambda_i\}_{i=1}^\infty \subset [0, \infty)$ satisfy for each integer $i \geq 1$,

$$(x_i, \lambda_i) \in A(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau\in\mathcal{E}}, \delta).$$

By Lemma 5.9 and the choice of δ, for each integer $i \geq 0$,

$$\|B_2(x_{i+1} - B_1(\sum_{\tau\in\mathcal{E}} P_{\Omega_{i+1,\tau}, w_{i+1,\tau}} \circ \pi_\tau)(x_i))\|$$

$$\leq m_0^{1/2}\delta(\bar{q}+1)\mathrm{Card}(\mathcal{E}) \leq 4^{-1}\epsilon_0 Q^{-1}(2\bar{N}+1)^{-1}.$$

Theorem 2.9 implies that for all integers $i \geq Q$,

$$B(x_i, \epsilon_0/4) \cap F \neq \emptyset$$

and

$$d_X(x_i, C_s) < \epsilon_0,\ s = 1, \ldots, m.$$

Theorem 5.6 is proved.

Chapter 6
Proximal Point Algorithm

In a Hilbert space, we study the convergence of an iterative proximal point method to a common zero of a finite family of maximal monotone operators under the presence of perturbations. We show that the inexact proximal point method generates an approximate solution if perturbations are summable. We also show that if the perturbations are sufficiently small, then the inexact proximal point method produces approximate solutions.

6.1 Preliminaries and Main Results

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ which induces the norm $\| \cdot \|$.

A multifunction $T : X \to 2^X$ is called a monotone operator if and only if

$$\langle z - z', w - w' \rangle \ \geq 0 \quad \forall z, z', w, w' \in X$$
$$\text{such that } w \in T(z) \text{ and } w' \in T(z'). \tag{6.1}$$

It is called maximal monotone if, in addition, the graph

$$\{(z, w) \in X \times X : \ w \in T(z)\}$$

is not properly contained in the graph of any other monotone operator $T' : X \to 2^X$. A fundamental problem consists in determining an element z such that $0 \in T(z)$. For example, if T is the subdifferential ∂f of a lower semicontinuous convex function $f : X \to (-\infty, \infty]$, which is not identically infinity, then T is maximal monotone (see [93, 95]), and the relation $0 \in T(z)$ means that z is a minimizer of f.

Let $T : X \to 2^X$ be a maximal monotone operator. The proximal point algorithm generates, for any given sequence of positive real numbers and any starting point in

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the space, a sequence of points and the goal is to show the convergence of this sequence. Note that in a general infinite-dimensional Hilbert space this convergence is usually weak. The proximal algorithm for solving the inclusion $0 \in T(z)$ is based on the fact established by Minty [92], who showed that, for each $z \in X$ and each $c > 0$, there is a unique $u \in X$ such that

$$z \in (I + cT)(u),$$

where $I : X \to X$ is the identity operator ($Ix = x$ for all $x \in X$).

The operator

$$P_{c,T} := (I + cT)^{-1} \tag{6.2}$$

is therefore single-valued from all of X onto X (where c is any positive number). It is also nonexpansive:

$$\|P_{c,T}(z) - P_{c,T}(z')\| \le \|z - z'\| \text{ for all } z, z' \in X \tag{6.3}$$

and

$$P_{c,T}(z) = z \text{ if and only if } 0 \in T(z). \tag{6.4}$$

Following the terminology of Moreau [95] $P_{c,T}$ is called the proximal mapping associated with cT.

The proximal point algorithm generates, for any given sequence $\{c_k\}_{k=0}^{\infty}$ of positive real numbers and any starting point $z^0 \in X$, a sequence $\{z^k\}_{k=0}^{\infty} \subset X$, where

$$z^{k+1} := P_{c_k,T}(z^k), \ k = 0, 1, \ldots$$

It is not difficult to see that the

$$\text{graph}(T) := \{(x, w) \in X \times X : \ w \in T(x)\}$$

is closed in the norm topology of $X \times X$.

Set

$$F(T) = \{z \in X : \ 0 \in T(z)\}. \tag{6.5}$$

Usually algorithms considering in the literature generate sequences which converge weakly to an element of $F(T)$. In this chapter, for a given $\epsilon > 0$, we are interested to find a point x for which there is $y \in T(x)$ such that $\|y\| \le \epsilon$. This point x is considered as an ϵ-approximate solution.

For every point $x \in X$ and every nonempty set $A \subset X$ define

$$d(x, A) := \inf\{\|x - y\| : \ y \in A\}.$$

For every point $x \in X$ and every positive number r put

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

We denote by $\mathrm{Card}(A)$ the cardinality of the set A.

We apply the proximal point algorithm in order to obtain a good approximation of a point which is a common zero of a finite family of maximal monotone operators and a common fixed point of a finite family of quasi-nonexpansive operators.

Let $\mathcal{L}_1$ be a finite set of maximal monotone operators $T : X \to 2^X$ and $\mathcal{L}_2$ be a finite set of mappings $T : X \to X$. We suppose that the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is nonempty. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

Let $\bar{c} \in (0, 1]$ and let $\bar{c} = 1$, if $\mathcal{L}_2 = \emptyset$.

We suppose that

$$F(T) \ne \emptyset \text{ for any } T \in \mathcal{L}_1 \tag{6.6}$$

and that for every mapping $T \in \mathcal{L}_2$,

$$\mathrm{Fix}(T) := \{z \in X : \ T(z) = z\} \ne \emptyset, \tag{6.7}$$

$$\|z - x\|^2 \ge \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2$$
$$\text{for all } x \in X \text{ and all } z \in \mathrm{Fix}(T). \tag{6.8}$$

Let $\bar{\lambda} > 0$ and let $\bar{\lambda} = \infty$ and $\bar{\lambda}^{-1} = 0$, if $\mathcal{L}_1 = \emptyset$. Let a natural number

$$l \ge \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2). \tag{6.9}$$

Denote by $\mathcal{R}$ the set of all mappings

$$S : \{0, 1, 2, \dots\} \to \mathcal{L}_2 \cup \{P_{c,T} : \ T \in \mathcal{L}_1, \ c \in [\bar{\lambda}, \infty)\}$$

such that the following properties hold:

(P1) for every nonnegative integer p and every mapping $T \in \mathcal{L}_2$ there exists an integer $i \in \{p, \dots, p + l - 1\}$ satisfying $S(i) = T$;

(P2) for every nonnegative integer p and every monotone operator $T \in \mathcal{L}_1$ there exist an integer $i \in \{p, \dots, p + l - 1\}$ and a number $c \ge \bar{\lambda}$ satisfying $S(i) = P_{c,T}$.

Suppose that

$$F := (\cap_{T \in \mathcal{L}_1} F(T)) \cap (\cap_{Q \in \mathcal{L}_2} \mathrm{Fix}(Q)) \ne \emptyset. \tag{6.10}$$

Let $\epsilon > 0$. For every monotone operator $T \in \mathcal{L}_1$ define

$$F_\epsilon(T) = \{x \in X :\ T(x) \cap B(0, \epsilon) \neq \emptyset\} \tag{6.11}$$

and for every mapping $T \in \mathcal{L}_2$ set

$$\mathrm{Fix}_\epsilon(T) = \{x \in X :\ \|T(x) - x\| \leq \epsilon\}. \tag{6.12}$$

Define

$$F_\epsilon = (\cap_{T\in\mathcal{L}_1} F_\epsilon(T)) \cap (\cap_{Q\in\mathcal{L}_2} \mathrm{Fix}_\epsilon(Q)), \tag{6.13}$$

$$\tilde{F}_\epsilon = (\cap_{T\in\mathcal{L}_1} \{x \in X :\ d(x, F_\epsilon(T)) \leq \epsilon\})$$
$$\cap(\cap_{Q\in\mathcal{L}_2} \{x \in X :\ d(x, \mathrm{Fix}_\epsilon(Q)) \leq \epsilon\}). \tag{6.14}$$

We are interested to find solutions of the inclusion $x \in F$. In order to meet this goal we apply algorithms generated by mappings $S \in \mathcal{R}$. More precisely, we associate with every mapping $S \in \mathcal{R}$ the algorithm which generates, for every starting point $x_0 \in X$, a sequence of points $\{x_k\}_{k=0}^\infty \subset X$ such that

$$x_{k+1} := [S(k)](x_k),\ k = 0, 1, \ldots.$$

According to the results known in the literature, this sequence should converge weakly to a point of the set F. In this chapter, we study the behavior of the sequences generated by mappings $S \in \mathcal{R}$ taking into account perturbations.

In this chapter we prove the following three results: Theorem 6.1 which shows that the inexact proximal point method generates approximate solutions if perturbations are summable, Theorem 6.2 which establishes that the exact proximal point method generates approximate solutions, and Theorem 6.3 which demonstrates that the inexact proximal point method generates approximate solutions if the perturbations are small enough.

Theorem 6.1 *Assume that* $M > 0$*,*

$$B(0, M) \cap F \neq \emptyset, \tag{6.15}$$

$\epsilon \in (0, 1)$ *and that a sequence* $\{\epsilon_i\}_{i=1}^\infty \subset [0, \infty)$ *satisfy*

$$\Lambda := \sum_{i=1}^\infty \epsilon_i < \infty. \tag{6.16}$$

Let a natural number n_0 *be such that for each integer* $i \geq n_0$*,*

$$\epsilon_i < \min\{\epsilon(2l + 1)^{-1}, \epsilon\bar{\lambda}\}. \tag{6.17}$$

Assume that

$$S \in \mathcal{R},\ \{x_k\}_{k=0}^{\infty} \subset X,\ \|x_0\| \le M,$$
$$\|x_{k+1} - [S(k)](x_k)\| \le \epsilon_{k+1},\ k = 0, 1, \ldots. \tag{6.18}$$

Then

$$Card(\{i \in \{0, 1, \ldots\} :\ x_i \notin \tilde{F}_\epsilon\})$$
$$\le n_0 + ((2M + \Lambda)^2 + 2\bar{c}^{-1} l \Lambda (2M + \Lambda))(\min\{\epsilon(2l + 1)^{-1}, \epsilon\bar{\lambda}\})^{-2}.$$

Theorem 6.2 *Suppose that for every mapping* $T \in \mathcal{L}_2$*,*

$$\|T(y_1) - T(y_2)\| \le \|y_1 - y_2\| \textit{ for all } y_1, y_2 \in X. \tag{6.19}$$

Let $M > 1$*,*

$$B(0, M) \cap F \ne \emptyset, \tag{6.20}$$

$\epsilon \in (0, 1]$*,*

$$S \in \mathcal{R} \tag{6.21}$$

satisfies

$$S(k + l) = S(k) \textit{ for all integers } k \ge 0, \tag{6.22}$$

$\{x_k\}_{k=0}^{\infty} \subset X$ *satisfies*

$$\|x_0\| \le M, \tag{6.23}$$
$$x_{k+1} = [S(k)](x_k),\ k = 0, 1, \ldots. \tag{6.24}$$

Then for every integer

$$i \ge l(1 + 4M^2\bar{c}^{-1}(\min\{(4Ml)^{-1}(\epsilon\bar{\lambda})^2\bar{c},\ (4M)^{-1}\bar{c}\epsilon^2 l^{-3}\}))$$

the inclusion

$$x_i \in \tilde{F}_\epsilon$$

holds.

Theorem 6.3 *Suppose that for every mapping* $T \in \mathcal{L}_2$*,*

$$\|T(y_1) - T(y_2)\| \le \|y_1 - y_2\| \textit{ for all } y_1, y_2 \in X,$$

$r_0 \in (0, 1)$, $M_0 > 1$,

$$F_{r_0} \subset B(0, M_0), \tag{6.25}$$

$\epsilon_0 \in (0, r_0]$,

$$Q_0 = \lfloor l(1 + 4M_0^2 \bar{c}^{-1}(\min\{(4M_0 l)^{-1}(\epsilon_0 \bar{\lambda}/4)^2 \bar{c},\ (4M_0)^{-1}\bar{c}(\epsilon_0/4)^2 l^{-3}\})^{-2}))\rfloor, \tag{6.26}$$

$$\delta \in (0, 4^{-1}\epsilon_0(Q_0(2l+1))^{-1}]. \tag{6.27}$$

Assume that

$$S \in \mathcal{R}$$

satisfies

$$S(k + l) = S(k) \text{ for all integers } k \geq 0, \tag{6.28}$$

$\{x_k\}_{k=0}^{\infty} \subset X$ *satisfies*

$$\|x_0\| \leq M, \tag{6.29}$$

$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta,\ k = 0, 1, \ldots. \tag{6.30}$$

Then for every integer $i \geq Q_0$,

$$x_i \in \tilde{F}_{\epsilon_0}. \tag{6.31}$$

6.2 Auxiliary Results

It is easy to see that the following lemma holds.

Lemma 6.4 *Let* $z, x_0, x_1 \in X$. *Then*

$$2^{-1}\|z - x_0\|^2 - 2^{-1}\|z - x_1\|^2 - 2^{-1}\|x_0 - x_1\|^2 = \langle x_0 - x_1, x_1 - z\rangle.$$

Lemma 6.5 (Lemma 8.17 of [124]) *Assume that* $S \in \mathcal{R}$,

$$z \in F, \tag{6.32}$$

the integers p, q *satisfy* $0 \leq p < q$,

$$\{\epsilon_k\}_{k=p}^{q-1} \subset (0, \infty),\ \{x_k\}_{k=p}^{q} \subset X$$

and that for all integers $k \in \{p, \dots, q-1\}$*,*

$$\|x_{k+1} - [S(k)](x_k)\| \le \epsilon_{k+1}. \tag{6.33}$$

Then, for every integer $k \in \{p+1 \dots, q\}$ *the following inequality holds:*

$$\|z - x_k\| \le \|z - x_p\| + \sum_{i=p+1}^{k} \epsilon_i.$$

Proof Let an integer $k \in \{p, \dots, q-1\}$. By (6.3)–(6.5), (6.7), (6.8), (6.10), (6.32), and (6.33),

$$\|z - x_{k+1}\| \le \|z - [S(k)](x_k)\| + \|S(k)(x_k) - x_{k+1}\| \le \|z - x_k\| + \epsilon_{k+1}.$$

This implies the validity of Lemma 6.4. □

Lemma 6.6 (Lemma 8.18 of [124]) *Assume that for every mapping* $T \in \mathcal{L}_2$

$$\|T(x) - T(y)\| \le \|x - y\| \text{ for all } x, y \in X, \tag{6.34}$$

$S \in \mathcal{R}$*, the integers* p, q *satisfy* $0 \le p < q$*,*

$$\{\epsilon_k\}_{k=p+1}^{q} \subset (0, \infty),\ \{x_k\}_{k=p}^{q} \subset X,\ \{y_k\}_{k=p}^{q} \subset X,\ y_p = x_p$$

and that for all integers $k \in \{p, \dots, q-1\}$*,*

$$y_{k+1} = [S(k)](y_k),\ \|x_{k+1} - [S(k)](x_k)\| \le \epsilon_{k+1}. \tag{6.35}$$

Then, for every integer $k \in \{p+1 \dots, q\}$ *the following inequality holds:*

$$\|y_k - x_k\| \le \sum_{i=p+1}^{k} \epsilon_i. \tag{6.36}$$

Proof We prove the lemma by induction. In view of (6.35) and the equality $x_p = y_p$ inequality (6.36) holds for $k = p+1$.

Assume that an integer j satisfies $p+1 \le j \le q$, (6.36) holds for all $k = p+1, \dots, j$ and that $j < q$.

By (6.3), (6.34), (6.35), and (6.36) with $k = j$,

$$\|y_{j+1} - x_{j+1}\| \le \|[S(j)](y_j) - x_{j+1}\|$$
$$\le \|[S(j)](y_j) - [S(j)](x_j)\| + \|[S_j](x_j) - x_{j+1}\|$$
$$\le \|y_j - x_j\| + \epsilon_{j+1} \le \sum_{i=p+1}^{j} \epsilon_i + \epsilon_{j+1} = \sum_{i=p+1}^{j+1} \epsilon_i$$

and (6.36) holds for all $k = p+1, \ldots, j+1$. Therefore we showed by induction that (6.36) holds for all $k = p+1, \ldots, q$. This completes the proof of Lemma 6.6.

Lemma 6.7 (Lemma 8.19 of [124]) *Let*

$$A \in \mathcal{L}_2 \cup \{P_{c,T} : T \in \mathcal{L}_1, \ c \in [\bar{\lambda}, \infty)\}, \ x \in X, \\ z \in F. \tag{6.37}$$

Then

$$\|z - x\|^2 - \|z - A(x)\|^2 - \bar{c}\|x - A(x)\|^2 \geq 0. \tag{6.38}$$

Proof There are two cases:

(i) $T \in \mathcal{L}_2$;
(ii) there exist a mapping $T \in \mathcal{L}_1$,a number $c \in [\bar{\lambda}, \infty)$ such that $A = P_{c,T}$.

If (i) holds, then (6.38) follows from (6.8) and (6.37). Assume that (ii) holds. Then by Lemma 6.4,

$$2^{-1}\|z-x\|^2 - 2^{-1}\|z-A(x)\|^2 - 2^{-1}\|x-A(x)\|^2 = \langle x-A(x), A(x)-z\rangle. \tag{6.39}$$

By (ii) and (6.2),

$$A(x) = P_{c,T}(x) \text{ and } x \in (I + cT)(A(x)), \\ x - A(x) \in cT(A(x)). \tag{6.40}$$

By (6.1), (6.5), (6.10), (6.37)–(6.39), and (6.40), equation (6.38) holds. Lemma 6.7 is proved.

6.3 Proof of Theorem 6.1

In view of (6.15), there exists a point

$$z \in B(0, M) \cap F. \tag{6.41}$$

By (6.18) and (6.41),

$$\|z - x_0\| \leq 2M. \tag{6.42}$$

Set

$$\epsilon_0 = 0. \tag{6.43}$$

It follows from Lemma 6.5, (6.18), (6.18), and (6.41)–(6.43) that for all integers $i \geq 0$,

$$\|z - x_i\| \leq 2M + \sum_{j=0}^{i} \epsilon_i. \tag{6.44}$$

Put

$$\gamma_0 = \min\{\epsilon(2l+1)^{-1},\ \epsilon\bar{\lambda}\}. \tag{6.45}$$

In view of (6.17) and (6.45), for all integers $i \geq n_0$,

$$\epsilon_i < \gamma_0. \tag{6.46}$$

Let $i \geq 0$ be an integer. Lemma 6.7, (6.18), and (6.41) imply that

$$\|z - x_i\|^2 - \|z - [S(i)](x_i)\|^2 \geq \bar{c}\|x_i - [S(i)](x_i)\|^2. \tag{6.47}$$

By (6.16), (6.18), and (6.44),

$$\begin{aligned} &|\|z - x_{i+1}\|^2 - \|z - [S(i)](x_i)\|^2| \\ &\leq |\|z - x_{i+1}\| - \|z - [S(i)](x_i)\||(\|z - x_{i+1}\| + \|z - x_i\|) \\ &\leq 2\|x_{i+1} - [S(i)](x_i)\|(2M + \Lambda) \\ &\leq 2\epsilon_{i+1}(2M + \Lambda). \end{aligned} \tag{6.48}$$

It follows from (6.47) and (6.48) that

$$\begin{aligned} &\bar{c}\|x_i - [S(i)](x_i)\|^2 \\ &\leq \|z - x_i\|^2 - \|z - [S(i)](x_i)\|^2 \\ &\leq \|z - x_i\|^2 - \|z - x_{i+1}\|^2 + 2\epsilon_{i+1}(2M + \Lambda). \end{aligned} \tag{6.49}$$

By (6.16), (6.43), (6.44), and (6.49), for each natural number $n > n_0$,

$$\begin{aligned} (2M + \Lambda)^2 &\geq \|x_{n_0} - z\|^2 \\ &\geq \|x_{n_0} - z\|^2 - \|x_n - z\|^2 \\ &= \sum_{i=n_0}^{n-1} (\|x_i - z\|^2 - \|x_{i+1} - z\|^2) \end{aligned}$$

$$\geq \sum_{i=n_0}^{n-1} (\bar{c}\|x_i - [S(i)](x_i)\|^2 - 2\epsilon_{i+1}(2M + \Lambda))$$

and

$$(2M + \Lambda)^2 + 2(2M + \Lambda)\Lambda$$
$$= (2M + \Lambda)^2 + 2(2M + \Lambda)\sum_{j=0}^{\infty} \epsilon_j$$
$$\geq \sum_{i=n_0}^{n-1} (\bar{c}\|x_i - [S(i)](x_i)\|^2)$$
$$\geq \bar{c}\gamma_0^2 \mathrm{Card}(\{i \in \{n_0, \dots, n-1\} : \|x_i - [S(i)](x_i)\| \geq \gamma_0\}).$$

Since the relation above holds for every natural number $n > n_0$ we conclude that

$$\mathrm{Card}(\{i \in \{n_0, n_0 + 1, \dots\} : \|x_i - [S(i)](x_i)\| \geq \gamma_0\})$$
$$\leq \bar{c}^{-1}\gamma_0^{-2}((2M + \Lambda)^2 + 2(2M + \Lambda)\Lambda). \tag{6.50}$$

Define

$$E_0 = \{k \in \{n_0, n_0 + 1, \dots, \} : \|x_k - [S(k)](x_k)\| \geq \gamma_0\}. \tag{6.51}$$

In view of (6.50) and (6.51), we have

$$\mathrm{Card}(E_0) \leq \bar{c}^{-1}\gamma_0^{-2}((2M + \Lambda)^2 + 2(2M + \Lambda)\Lambda). \tag{6.52}$$

Define

$$E_1 = \{k \in \{n_0, n_0 + 1, \dots\} : [k, k + l - 1] \cap E_0 \neq \emptyset\}. \tag{6.53}$$

By (6.52) and (6.53), we have

$$\mathrm{Card}(E_1) \leq l\mathrm{Card}(E_0)$$
$$\leq l\bar{c}^{-1}\gamma_0^{-2}((2M + \Lambda)^2 + 2(2M + \Lambda)\Lambda). \tag{6.54}$$

Let an integer p satisfies

$$p \geq n_0, \ \ p \notin E_1. \tag{6.55}$$

In view of (6.51), (6.53), and (6.55),

$$\{p, \dots, p + l - 1\} \cap E_0 = \emptyset \tag{6.56}$$

and for each $k \in \{p, \dots, p+l-1\}$,

$$\|x_k - [S(k)](x_k)\| < \gamma_0. \tag{6.57}$$

By (6.18), (6.46), (6.55), and (6.57), for each $k \in \{p, \dots, p+l-1\}$,

$$\begin{aligned} \|x_k - x_{k+1}\| &\le \|x_k - [S(k)](x_k)\| + \|[S(k)](x_k) - x_{k+1}\| \\ &< \gamma_0 + \epsilon_{k+1} < 2\gamma_0. \end{aligned} \tag{6.58}$$

In view of (6.58), for all integers $k_1, k_2 \in \{p, \dots, p+l\}$, we have

$$\|x_{k_1} - x_{k_2}\| \le 2l\gamma_0. \tag{6.59}$$

Let $T \in \mathcal{L}_2$. It follows from property (P1) that there exists an integer $i \in \{p, \dots, p+l-1\}$ such that

$$S(i) = T. \tag{6.60}$$

In view of (6.57) and (6.60),

$$\|x_i - T(x_i)\| < \gamma_0. \tag{6.61}$$

By (6.45), (6.59), and (6.61), we have

$$d(x_p, \mathrm{Fix}_\epsilon(T)) \le \|x_p - x_i\| \le 2l\gamma_0 \le \epsilon. \tag{6.62}$$

Relation (6.62) implies that

$$d(x_p, \mathrm{Fix}_\epsilon(T)) \le \epsilon \text{ for all } T \in \mathcal{L}_2. \tag{6.63}$$

Let $T \in \mathcal{L}_1$. Property (P2) implies that there exist

$$i \in \{p, \dots, p+l-1\}, \ c \ge \bar{\lambda} \tag{6.64}$$

such that

$$S(i) = P_{c,T}. \tag{6.65}$$

By (6.57) and (6.65),

$$\|x_i - P_{c,T}(x_i)\| < \gamma_0. \tag{6.66}$$

It follows from (6.2) that

$$x_i \in (I + cT)(P_{c,T}(x_i)),$$
$$x_i - P_{c,T}(x_i) \in cT(P_{c,T}(x_i)),$$
$$c^{-1}(x_i - P_{c,T}(x_i)) \in T(P_{c,T}(x_i)). \tag{6.67}$$

By (6.45), (6.64), and (6.66), we have

$$\|c^{-1}(x_i - P_{c,T}(x_i))\| \le c^{-1}\gamma_0 \le \bar{\lambda}^{-1}\gamma_0 \le \epsilon. \tag{6.68}$$

Relations (6.11), (6.67), and (6.68) imply that

$$P_{c,T}(x_i) \in F_\epsilon(T). \tag{6.69}$$

It follows from (6.45), (6.59), (6.64), and (6.66) that

$$\|x_p - P_{c,T}(x_i)\| \le \|x_p - x_i\| + \|x_i - P_{c,T}(x_i)\|$$
$$\le 2l\gamma_0 + \gamma_0 \le \epsilon. \tag{6.70}$$

In view of (6.69) and (6.70),

$$d(x_p, F_\epsilon(T)) \le \|x_p - P_{c,T}(x_i)\| \le \epsilon.$$

Therefore by (6.70),

$$d(x_p, F_\epsilon(T)) \le \epsilon \text{ for all } T \in \mathcal{L}_1. \tag{6.71}$$

By (6.63) and (6.71),

$$x_p \in \tilde{F}_\epsilon$$

for all integers $p \ge n_0$ satisfying $p \notin E_1$. Together with (6.45) and (6.54) this implies that

$$\text{Card}(\{i \in \{0, 1, \dots\} : \ x_i \notin \tilde{F}_\epsilon\})$$
$$\le n_0 + \text{Card}(E_1)$$
$$\le n_0 + ((2M + \Lambda)^2 + 2\Lambda(2M + \Lambda))\bar{c}^{-1}l(\min\{\epsilon(2l + 1)^{-1}, \epsilon\bar{\lambda}\})^{-2}.$$

Theorem 6.1 is proved.

6.4 Proof of Theorem 6.2

Put

$$\gamma_0 = \min\{(4Ml)^{-1}(\epsilon\bar{\lambda})^2\bar{c},\ (4M)^{-1}\bar{c}\epsilon^2 l^{-3}\}. \tag{6.72}$$

In view of (6.24), for each integer $i \geq 0$,

$$x_{i+1} = [S(i)](x_i). \tag{6.73}$$

Set

$$T_S = S(l-1)\cdots S(0) = \prod_{i=0}^{l-1} S(i). \tag{6.74}$$

It follows from (6.22), (6.73), and (6.74) that for each integer $i \geq 0$,

$$x_{(i+1)l} = S((i+1)l-1)\cdots S(il)(x_{il}) = T_S(x_{il}). \tag{6.75}$$

In view of (6.20), there exists a point

$$z \in B(0, M) \cap F. \tag{6.76}$$

By (6.23) and (6.76),

$$\|z - x_0\| \leq 2M. \tag{6.77}$$

Lemma 6.7, (6.73), (6.76), and (6.77) imply that for all integers $k \geq 0$,

$$\|z - x_k\|^2 - \|z - x_{k+1}\|^2 \geq \bar{c}\|x_k - x_{k+1}\|^2, \tag{6.78}$$

$$\|z - x_{k+1}\| \leq \|z - x_k\| \tag{6.79}$$

and

$$\|z - x_k\| \leq 2M. \tag{6.80}$$

Define

$$E_0 = \{k \in \{0, 1, \dots, \} :\ \max\{\|x_{i+1} - x_i\| : i = kl, \dots, (k+1)l - 1\} \geq \gamma_0\}. \tag{6.81}$$

Let n be a natural number. It follows from (6.77) and (6.78) that

$$\begin{aligned}4M^2 &\geq \|x_0 - z\|^2 \\ &\geq \|x_0 - z\|^2 - \|x_{nl} - z\|^2 \\ &= \sum_{k=0}^{n-1}(\|x_{kl} - z\|^2 - \|x_{(k+1)l} - z\|^2) \\ &= \sum_{k=0}^{n-1}[\sum_{j=kl}^{(k+1)l-1}(\|x_j - z\|^2 - \|x_{j+1} - z\|^2)] \\ &\geq \sum_{k=0}^{n-1}(\bar{c}\sum_{j=kl}^{(k+1)l-1}\|x_j - x_{j+1}\|^2) \\ &\geq \bar{c}\gamma_0^2 \mathrm{Card}(\{k \in \{0, \dots, n-1\} : \max\{\|x_{j+1} - x_j\| : \\ &\qquad j = kl, \dots, (k+1)l - 1\} \geq \gamma_0\})\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Card}(\{k \in \{0, \dots, n-1\} : \max\{\|x_{j+1} - x_j\| : \\ &\qquad j = kl, \dots, (k+1)l - 1\} \geq \gamma_0\}) \\ &\qquad \leq 4M^2\bar{c}^{-1}\gamma_0^{-2}.\end{aligned}$$

Since the relation above holds for every natural number n we conclude that

$$\begin{aligned}&\mathrm{Card}(\{k \in \{0, 1, \dots\} : \max\{\|x_{j+1} - x_j\| : \\ &\qquad j = kl, \dots, (k+1)l - 1\} \geq \gamma_0\}) \\ &\qquad \leq 4M^2\bar{c}^{-1}\gamma_0^{-2}.\end{aligned} \tag{6.82}$$

In view of (6.81) and (6.82), we have

$$\mathrm{Card}(E_0) \leq 4M^2\bar{c}^{-1}\gamma_0^{-2}. \tag{6.83}$$

In view of (6.82), there exists an integer $q_0 \geq 0$ such that

$$q_0 \leq 4M^2\bar{c}^{-1}\gamma_0^{-2} + 1, \tag{6.84}$$

$$\|x_j - x_{j+1}\| < \gamma_0, \ \ j = q_0 l, \dots, (q_0 + 1)l - 1. \tag{6.85}$$

By (6.85),

$$\|x_{q_0 l} - x_{(q_0+1)}\| \leq \gamma_0 l. \tag{6.86}$$

It follows from (6.3), (6.19), (6.74), (6.75), and (6.86) that for every integer $q > q_0$,

$$\begin{aligned} &\|x_{ql} - x_{(q+1)l}\| \\ &= \|T_S^{q-q_0}(x_{q_0 l}) - T_S^{q-q_0}(x_{(q_0+1)l})\| \\ &\le \|x_{q_0 l} - x_{(q_0+1)l}\| \le \gamma_0 l. \end{aligned} \tag{6.87}$$

Let $q \ge q_0$ be an integer. Relations (6.86) and (6.87) imply that

$$\|x_{ql} - x_{(q+1)l}\| \le \gamma_0 l. \tag{6.88}$$

By (6.78), (6.80), and (6.88),

$$\begin{aligned} \gamma_0 l &\ge \|x_{ql} - x_{(q+1)l}\| \\ &\ge \|z - x_{ql}\| - \|z - x_{(q+1)l}\| \\ &\ge (\|z - x_{ql}\|^2 - \|z - x_{(q+1)l}\|^2)(4M)^{-1}, \\ 4M\gamma_0 l &\ge \|z - x_{ql}\|^2 - \|z - x_{(q+1)l}\|^2 \\ &= \sum_{i=ql}^{(q+1)l-1} [\|z - x_i\|^2 - \|z - x_{i+1}\|^2] \\ &\ge \sum_{i=ql}^{(q+1)l-1} \bar{c}\|x_i - x_{i+1}\|^2 \end{aligned}$$

and for each $i = ql, \dots, (q+1)l - 1$,

$$\|x_i - x_{i+1}\| \le (4M\gamma_0 l\bar{c}^{-1})^{1/2}. \tag{6.89}$$

In view of (6.89), for each $i, j \in \{ql, \dots, (q+1)l\}$,

$$\|x_i - x_j\| \le l(4M\gamma_0 l\bar{c}^{-1})^{1/2}. \tag{6.90}$$

Let $T \in \mathcal{L}_2$. It follows from property (P1) that there exists an integer $j \in \{ql, \dots, (q+l)l - 1\}$ such that

$$S(j) = T. \tag{6.91}$$

In view of (6.73), (6.89), and (6.91),

$$\|x_j - T(x_j)\| = \|x_j - S(j)(x_j)\| = \|x_{j+1} - x_j\| \le (4M\gamma_0 l\bar{c}^{-1})^{1/2}. \tag{6.92}$$

By (6.72), (6.90), and (6.92), for each $i \in \{ql, \ldots, (q+1)l\}$,

$$d(x_i, \mathrm{Fix}_{(4M\gamma_0 l\bar{c}^{-1})^{1/2}}(T)) \le l(4M\gamma_0 l\bar{c}^{-1})^{1/2} \le \epsilon$$

and

$$d(x_i, \mathrm{Fix}_\epsilon(T)) \le \epsilon \text{ for all } T \in \mathcal{L}_2. \tag{6.93}$$

Let $T \in \mathcal{L}_1$. Property (P2) implies that there exist

$$j \in \{ql, \ldots, (q+l)l - 1\},\ c \ge \bar{\lambda} \tag{6.94}$$

such that

$$S(j) = P_{c,T}. \tag{6.95}$$

By (6.73), (6.89), (6.94), and (6.95),

$$\|x_j - P_{c,T}(x_j)\| - \|x_j - S(j)(x_j)\| = \|x_j - x_{j+1}\| \le (4M\gamma_0 l\bar{c}^{-1})^{1/2}. \tag{6.96}$$

It follows from (6.2) that

$$x_j \in (I + cT)(P_{c,T}(x_j)),$$
$$x_j - P_{c,T}(x_j) \in cT(P_{c,T}(x_j)),$$
$$c^{-1}(x_j - P_{c,T}(x_j)) \in T(P_{c,T}(x_j)). \tag{6.97}$$

By (6.72), (6.94), and (6.96), we have

$$\|c^{-1}(x_j - P_{c,T}(x_j))\|$$
$$\le c^{-1}(4M\gamma_0 l\bar{c}^{-1})^{1/2} \le \bar{\lambda}^{-1}(4M\gamma_0 l\bar{c}^{-1})^{1/2} \le \epsilon. \tag{6.98}$$

Relations (6.11), (6.73), (6.95), (6.97), and (6.98) imply that

$$x_{j+1} = P_{c,T}(x_j) \in F_\epsilon(T). \tag{6.99}$$

It follows from (6.72), (6.90), (6.94), and (6.99) that for each $i = ql, \ldots, (q+1)l$,

$$d(x_i, F_\epsilon(T)) \le l(4M\gamma_0 l\bar{c}^{-1})^{1/2} \le \epsilon$$

for all $T \in \mathcal{L}_1$. Together with (6.14) and (6.93) this implies that

$$x_i \in \tilde{F}_\epsilon$$

for all integers $i \in \{ql, \ldots, (q+1)l\}$ and all integers $q \ge q_0$. Theorem 6.2 is proved.

6.5 Proof of Theorem 6.3

Theorem 6.3 is deduced from Theorems 2.9 and 6.2. Let $Y = X$, $N = l$, $\rho(y, z) = \|y - z\|$, $y, z \in X$, $\mathfrak{A}$ be the set of all mappings

$$S : \{1, 2, \dots\} \to \mathcal{L}_2 \cup \{P_{c,T} : \ T \in \mathcal{L}_1, \ c \in [\bar{\lambda}, \infty)\}$$

such that the mapping

$$i \to S(i + 1), \ i = 0, 1, \dots$$

have properties (P1) and (P2) and that

$$S(k + l) = S(l) \text{ for all integers } k \geq 1.$$

Set

$$F = \tilde{F}_{\epsilon_0/4}.$$

For each $S \in \mathfrak{A}$ define

$$\widehat{S} : \{0, 1, 2, \dots\} \to \mathcal{L}_2 \cup \{P_{c,T} : \ T \in \mathcal{L}_1, \ c \in [\bar{\lambda}, \infty)\}$$

by

$$\widehat{S}(i) = S(i + 1) \text{ for all integers } i \geq 0$$

and set

$$\widehat{\mathfrak{A}} = \{\widehat{S} : \ S \in \mathfrak{A}\}.$$

Theorem 6.2 implies that for every $M > 0$ property (P6) holds with

$$Q_0 = \lfloor l(1 + 4M^2\bar{c}^{-1}(\min\{(4Ml)^{-1}(\epsilon\bar{\lambda})^2\bar{c}, \ (4M)^{-1}\bar{c}\epsilon^2 l^{-3}\}))\rfloor.$$

In view of Theorems 2.9 and 6.2, for all integer $i \geq Q_0$,

$$B(x_i, \epsilon_0/4) \cap F \neq \emptyset.$$

By the relation above and the choice of F, for all integers $i \geq Q_0$,

$$x_i \in \tilde{F}_{\epsilon_0}.$$

Theorem 6.3 is proved.

Chapter 7
Dynamic String-Averaging Proximal Point Algorithm

In a Hilbert space, we study the convergence of a dynamic string-averaging proximal point method to a common zero of a finite family of maximal monotone operators under the presence of perturbations. Our main goal is to obtain an approximate solution of the problem using perturbed algorithms. We show that the inexact dynamic string-averaging proximal point algorithm generates an approximate solution if perturbations are summable. We also show that if the perturbations are sufficiently small, then the inexact produces approximate solutions.

7.1 Preliminaries and Main Results

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ which induces the complete norm $\| \cdot \|$.

For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) = \inf\{\|x - y\| : \ y \in A\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) := \{y \in X : \ \|x - y\| \leq r\}.$$

Denote by $\mathrm{Card}(A)$ the cardinality of a set A. The sum over an empty set is assumed to be zero.

Recall (see Section 6.1) that a multifunction $T : X \to 2^X$ is called a monotone operator if

$$\langle z - z', w - w' \rangle \ \geq 0 \quad \forall z, z', w, w' \in X$$

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$$\text{such that } w \in T(z) \text{ and } w' \in T(z').$$

It is called maximal monotone if, in addition, the graph

$$\{(z, w) \in X \times X : \ w \in T(z)\}$$

is not properly contained in the graph of any other monotone operator $T' : X \to 2^X$.

Let $T : X \to 2^X$ be a maximal monotone operator. Then (see Section 6.1) for each $z \in X$ and each $c > 0$, there is a unique $u \in X$ such that

$$z \in (I + cT)(u),$$

where $I : X \to X$ is the identity operator ($Ix = x$ for all $x \in X$).

The operator

$$P_{c,T} := (I + cT)^{-1} \tag{7.1}$$

is therefore single-valued from all of X onto X (where c is any positive number). It is also nonexpansive:

$$\|P_{c,T}(z) - P_{c,T}(z')\| \le \|z - z'\| \text{ for all } z, z' \in X \tag{7.2}$$

and

$$P_{c,T}(z) = z \text{ if and only if } 0 \in T(z) \tag{7.3}$$

(see Section 6.1).

Set

$$F(T) = \{z \in X : \ 0 \in T(z)\}. \tag{7.4}$$

Let $\mathcal{L}_1$ be a finite set of maximal monotone operators $T : X \to 2^X$ and $\mathcal{L}_2$ be a finite set of mappings $T : X \to X$. We suppose that the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is nonempty. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

Let $\bar{c} \in (0, 1]$ and let $\bar{c} = 1$, if $\mathcal{L}_2 = \emptyset$.

We suppose that

$$F(T) \ne \emptyset \text{ for any } T \in \mathcal{L}_1 \tag{7.5}$$

and that for each $T \in \mathcal{L}_2$,

$$\text{Fix}(T) := \{z \in X : \ T(z) = z\} \ne \emptyset, \tag{7.6}$$

$$\|z - x\|^2 \ge \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2 \tag{7.7}$$

$$\text{for all } x \in X \text{ and all } z \in \text{Fix}(T).$$

Suppose that

$$F := (\cap_{T\in\mathcal{L}_1} F(T)) \cap (\cap_{Q\in\mathcal{L}_2} \mathrm{Fix}(Q)) \neq \emptyset. \tag{7.8}$$

Let $\epsilon > 0$. For any $T \in \mathcal{L}_1$ set

$$F_\epsilon(T) = \{x \in X :\ T(x) \cap B(0, \epsilon) \neq \emptyset\} \tag{7.9}$$

and for any $T \in \mathcal{L}_2$ put

$$\mathrm{Fix}_\epsilon(T) = \{x \in X :\ \|T(x) - x\| \leq \epsilon\}. \tag{7.10}$$

Set

$$\begin{aligned}\tilde{F}_\epsilon &= (\cap_{T\in\mathcal{L}_1}\{x \in X :\ d(x, F_\epsilon(T)) \leq \epsilon\}) \\ &\cap(\cap_{Q\in\mathcal{L}_2}\{x \in X :\ d(x, \mathrm{Fix}_\epsilon(Q)) \leq \epsilon\}).\end{aligned} \tag{7.11}$$

Let $\bar{\lambda} > 0$ and let $\bar{\lambda} = \infty$ and $\bar{\lambda}^{-1} = 0$, if $\mathcal{L}_1 = \emptyset$. Set

$$\mathcal{L} = \mathcal{L}_2 \cup \{P_{c,T} :\ T \in \mathcal{L}_1,\ c \in [\bar{\lambda}, \infty)\}. \tag{7.12}$$

Next we describe the dynamic string-averaging method with variable strings and weights.

By a mapping vector, we mean a vector $T = (T_1, \dots, T_p)$ such that $T_i \in \mathcal{L}$ for all $i = 1, \dots, p$.

For a mapping vector $T = (T_1, \dots, T_q)$ set

$$p(T) = q,\ P[T] = T_q \cdots T_1. \tag{7.13}$$

It is easy to see that for each mapping vector $T = (T_1, \dots, T_p)$,

$$P[T](x) = x \text{ for all } x \in F, \tag{7.14}$$

$$\|P[T](x) - P[T](y)\| = \|x - P[T](y)\| \leq \|x - y\| \tag{7.15}$$

for every $x \in F$ and every $y \in X$.

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of mapping vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{T\in\Omega} w(T) = 1. \tag{7.16}$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{T\in\Omega} w(T) P[T](x),\ x \in X. \tag{7.17}$$

It is not difficult to see that

$$P_{\Omega,w}(x) = x \text{ for all } x \in F, \tag{7.18}$$

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| = \|x - P_{\Omega,w}(y)\| \le \|x - y\| \tag{7.19}$$

for all $x \in F$ and all $y \in X$.

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm.

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k).$$

Fix a number

$$\Delta \in (0, \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2)^{-1}) \tag{7.20}$$

and natural numbers $\bar{N}$ and $\bar{q}$ satisfying

$$\bar{q} \ge \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2). \tag{7.21}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(T) \le \bar{q} \ \text{ for all } T \in \Omega, \tag{7.22}$$

$$w(T) \ge \Delta \text{ for all } T \in \Omega. \tag{7.23}$$

Denote by $\mathcal{R}$ the set of all sequences

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

such that the following properties hold:

(P1) for each integer $j \ge 1$ and each $S \in \mathcal{L}_2$ there exist $k \in \{j, \dots, j + \bar{N} - 1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_k$ such that

$$S \in \{T_1, \dots, T_{p(T)}\};$$

(P2) for each integer $j \ge 1$ and each $S \in \mathcal{L}_1$ there exist $k \in \{j, \dots, j + \bar{N} - 1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_k$ and $c \ge \bar{\lambda}$ such that

$$P_{c,S} \in \{T_1, \dots, T_{p(T)}\}.$$

In order to state our main results we need the following definitions.

Let $\delta \geq 0$, $x \in X$ and let $T = (T_1, \ldots, T_{p(t)})$ be a mapping vector. Define

$$A_0(x, T, \delta) = \{(y, \lambda) \in X \times R^1 : \text{ there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that}$$
$$y_0 = x \text{ and for all } i = 1, \ldots, p(t),$$
$$\|y_i - T_i(y_{i-1})\| \leq \delta,$$
$$y = y_{p(T)},$$
$$\lambda = \max\{\|y_i - y_{i-1}\| : i = 1, \ldots, p(T)\}\}. \tag{7.24}$$

Let $\delta \geq 0$, $x \in X$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$A(x, (\Omega, w), \delta) = \{(y, \lambda) \in X \times R^1 : \text{ there exist}$$
$$(y_T, \lambda_T) \in A_0(x, T, \delta),\ T \in \Omega \text{ such that}$$
$$\|y - \sum_{T \in \Omega} w(T) y_T\| \leq \delta,\ \lambda = \max\{\lambda_T : T \in \Omega\}\}. \tag{7.25}$$

In this chapter we prove the following three results: Theorem 7.1 which shows that the inexact dynamic string-averaging method generates approximate solutions if perturbations are summable, Theorem 7.2 which establishes that the exact dynamic string-averaging method generates approximate solutions, and Theorem 7.3 which demonstrates that the inexact dynamic string-averaging method generates approximate solutions if the perturbations are small enough.

Theorem 7.1 *Let $M > 0$, $\epsilon \in (0, 1]$,*

$$B(0, M) \cap F \neq \emptyset, \tag{7.26}$$
$$\{\epsilon_i\}_{i=1}^{\infty} \subset [0, \infty),$$
$$\Lambda := \sum_{i=1}^{\infty} < \infty \tag{7.27}$$

and n_0 be a natural number such that for all integers $i \geq n_0$,

$$\epsilon_i < \epsilon(\bar{N} + 1)^{-1}(1 + \bar{q})^{-1} \min\{\bar{\lambda}, 1\}. \tag{7.28}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \in \mathcal{R}, \tag{7.29}$$
$$x_0 \in B(0, M), \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{7.30}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \epsilon_i). \tag{7.31}$$

Then

$$Card(\{i \in \{0, 1, \dots\} : x_i \notin \tilde{F}_\epsilon\})$$
$$\le n_0 + \bar{N}\bar{c}^{-1}\Delta^{-1}[(2M + \Lambda(\bar{q} + 1)^2$$
$$+8\bar{q}\Lambda(2M + (\bar{q} + 1)\Lambda]\epsilon^{-2}(\bar{N} + 1)(\bar{q} + 1)(\min\{1, \bar{\lambda}\})^{-2}.$$

Theorem 7.2 *Assume that for each $T \in \mathcal{L}_2$ and each $x, y \in X$,*

$$\|T(x) - T(y)\| \le \|x - y\|. \tag{7.32}$$

Let $M > 1$, $\epsilon \in (0, 1)$,

$$B(0, M) \cap F \neq \emptyset, \tag{7.33}$$

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \in \mathcal{R}, \tag{7.34}$$

$$P_{\Omega_i, w_i} = P_{\Omega_{i+\bar{N}}, w_{i+\bar{N}}} \text{ for all integers } i \ge 0, \tag{7.35}$$

$$x_0 \in B(0, M),\ \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{7.36}$$

satisfy for each integer $i \ge 1$,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), 0). \tag{7.37}$$

Then for each integer

$$i \ge \bar{N}(1 + 64M^4\bar{c}^{-3}\Delta^{-3}\epsilon^{-2}(\bar{q} + 1)^4(\bar{N} + 1)^4 \min\{1, \bar{\lambda}\}^{-2}),$$
$$x_i \in F_\epsilon.$$

Theorem 7.3 *Assume that for each $T \in \mathcal{L}_2$ and each $x, y \in X$,*

$$\|T(x) - T(y)\| \le \|x - y\|. \tag{7.38}$$

Let $r_0 \in (0, 1)$, $M_0 > 1$,

$$F_{r_0} \subset B(0, M_0), \tag{7.39}$$

$\epsilon_0 \in (0, r_0)$,

$$Q_0 = \lfloor \bar{N}(1 + 64M^4\bar{c}^{-3}\Delta^{-3}\epsilon_0^{-2}(\bar{q} + 1)^4(\bar{N} + 1)^4 \min\{1, \bar{\lambda}\}^{-2})\rfloor \tag{7.40}$$

and

$$\delta = (\epsilon_0/4)(2\bar{N} + 1)^{-1}Q_0^{-1}(\bar{q} + 1)^{-1}.$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \in \mathcal{R},$$

$$P_{\Omega_i, w_i} = P_{\Omega_{i+\bar{N}}, w_{i+\bar{N}}} \text{ for all integers } i \geq 0,$$

$$x_0 \in B(0, M),\ \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each integer $i \geq 1$,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta).$$

Then for each integer $i \geq Q_0$,

$$x_i \in \tilde{F}_{\epsilon_0}.$$

7.2 Proof of Theorem 7.1

Set

$$\gamma_0 = \epsilon(\bar{N} + 1)^{-1}(\bar{q} + 1)^{-1} \min\{1, \bar{\lambda}\}. \tag{7.41}$$

By (7.26) there exists

$$z \in B(0, M) \cap F. \tag{7.42}$$

Let $k \geq 0$ be an integer. By (7.31),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, (\Omega_{k+1}, w_{k+1}), \epsilon_{k+1}). \tag{7.43}$$

By (7.25) and (7.43) there exist

$$(y_{k,T}, \lambda_{k,T}) \in A_0(x_k, T, \epsilon_{k+1}),\ T \in \Omega_{k+1} \tag{7.44}$$

such that

$$\|x_{k+1} - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\| \leq \epsilon_{k+1}, \tag{7.45}$$

$$\lambda_{k+1} = \max\{\lambda_{k,T} : T \in \Omega_{k+1}\}. \tag{7.46}$$

Let

$$T = (T_1, \ldots, T_{p(T)}) \in \Omega_{k+1}.$$

It follows from (7.24) and (7.44) that there exists a finite sequence

$$\{y_i^{(k,T)}\}_{i=0}^{p(T)} \subset X$$

such that

$$y_0^{(k,T)} = x_k, \tag{7.47}$$

$$\|y_i^{(k,T)} - T_i(y_{i-1}^{(k,T)})\| \leq \epsilon_{k+1} \text{ for each integer } i = 1, \dots, p(T), \tag{7.48}$$

$$y_{p(T)}^{(k,T)} = y_{k,T}, \tag{7.49}$$

$$\lambda_{k,T} = \max\{\|y_i^{(k,T)} - y_{i-1}^{(k,T)}\| : \ i = 1, \dots, p(T)\}. \tag{7.50}$$

By (7.7), (7.12), (7.29), (7.42), and Lemma 6.7, for each

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}$$

and each $i \in \{1, \dots, p(T)\}$, we have

$$\|z - y_{i-1}^{(k,T)}\|^2 \geq \|z - T_i(y_{i-1}^{(k,T)})\|^2 + \bar{c}\|y_{i-1}^{(k,T)} - T_i(y_{i-1}^{(k,T)})\|^2 \tag{7.51}$$

and

$$\|z - T_i(y_{i-1}^{(k,T)})\| \leq \|z - y_{i-1}^{(k,T)}\|. \tag{7.52}$$

Relations (7.48) and (7.52) imply that for each

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}$$

and each $i \in \{1, \dots, p(T)\}$, we have

$$\begin{aligned} \|z - y_i^{(k,T)}\| &\leq \|z - T_i(y_{i-1}^{(k,T)})\| + \|T_i(y_{i-1}^{(k,T)}) - y_i^{(k,T)}\| \\ &\leq \|z - y_{i-1}^{(k,T)}\| + \epsilon_{k+1}. \end{aligned} \tag{7.53}$$

In view of (7.22), (7.47), (7.49), and (7.53), for each

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}$$

and each $i \in \{0, 1, \dots, p(T)\}$,

$$\begin{aligned} \|z - y_i^{(k,T)}\| &\leq \|z - y_0^{(k,T)}\| + i\epsilon_{k+1} \\ &= \|z - x_k\| + i\epsilon_{k+1}, \end{aligned} \tag{7.54}$$

$$\|z - y_{k,T}\| = \|z - y^{(k,T)}_{p(T)}\|$$
$$\leq \|z - x_k\| + p(T)\epsilon_{k+1} \leq \|z - x_k\| + \bar{q}\epsilon_{k+1}. \tag{7.55}$$

By (7.16), (7.45), (7.55), and the convexity of the norm $\|\cdot\|$,

$$\|z - x_{k+1}\| \leq \|z - \sum_{T\in\Omega_{k+1}} w_{k+1}(T)y_{k,T}\| + \|\sum_{T\in\Omega_{k+1}} w_{k+1}(T)y_{k,T} - x_{k+1}\|$$
$$\leq \sum_{T\in\Omega_{k+1}} w_{k+1}(T)\|z - y_{k,T}\| + \epsilon_{k+1}$$
$$\leq \|z - x_k\| + (\bar{q}+1)\epsilon_{k+1}. \tag{7.56}$$

In view (7.30) and (7.42),

$$\|x_0 - z\| \leq 2M. \tag{7.57}$$

Set

$$\epsilon_0 = 0. \tag{7.58}$$

By (7.22), (7.54), and (7.55)–(7.58), for each integer $k \geq 0$, for each

$$T = (T_1, \ldots, T_{p(T)}) \in \Omega_{k+1}$$

and each $i = 1, \ldots, p(T)$,

$$\|z - x_k\| \leq \|z - x_0\| + (\bar{q}+1)\sum_{i=0}^{k}\epsilon_i$$
$$\leq 2M + (\bar{q}+1)\sum_{i=0}^{k}\epsilon_i, \tag{7.59}$$
$$\|z - y_{k,T}\| \leq \|z - x_k\| + \bar{q}\epsilon_{k+1}$$
$$\leq 2M + (\bar{q}+1)\sum_{i=0}^{k}\epsilon_i, \tag{7.60}$$
$$\|z - y_i^{(k,T)}\| \leq \|z - x_k\| + i\epsilon_{k+1}$$
$$\leq 2M + (\bar{q}+1)\sum_{i=0}^{k}\epsilon_i + \bar{q}\epsilon_{k+1}$$
$$\leq 2M + (\bar{q}+1)\sum_{i=0}^{k+1}\epsilon_i. \tag{7.61}$$

By (7.27) and (7.59)–(7.61), for each integer $k \ge 0$, for each

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}$$

and each $i = 1, \dots, p(T)$,

$$\|z - x_k\| \le 2M + \Lambda(\bar{q} + 1), \tag{7.62}$$

$$\|z - y_{k,T}\| \le 2M + \Lambda(\bar{q} + 1), \tag{7.63}$$

$$\|z - y_i^{(k,T)}\| \le 2M + \Lambda(\bar{q} + 1). \tag{7.64}$$

Let $k \ge 0$ be an integer,

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1},\ i \in \{1, \dots, p(T)\}.$$

By (7.48), (7.52), and (7.64),

$$\begin{aligned}
&| \|z - T_i(y_{i-1}^{(k,T)})\|^2 - \|z - y_i^{(k,T)}\|^2 | \\
&\le | \|z - T_i(y_{i-1}^{(k,T)})\| - \|z - y_i^{(k,T)}\| | (\|z - T_i(y_{i-1}^{(k,T)})\| + \|z - y_i^{(k,T)}\|) \\
&\le 2\|y_i^{(k,T)} - T_i(y_{i-1}^{(k,T)})\|(2M + \Lambda(\bar{q} + 1)) \\
&\le 2\epsilon_{k+1}(2M + \Lambda(\bar{q} + 1)).
\end{aligned} \tag{7.65}$$

In view of (7.51) and (7.65),

$$\begin{aligned}
&\|z - y_{i-1}^{(k,T)}\|^2 - \|z - y_i^{(k,T)}\|^2 \\
&\ge \|z - y_{i-1}^{(k,T)}\|^2 - \|z - T_i(y_{i-1}^{(k,T)})\|^2 - 2\epsilon_{k+1}(2M + \Lambda(\bar{q} + 1)) \\
&\ge \bar{c}\|y_{i-1}^{(k,T)} - T_i(y_{i-1}^{(k,T)})\|^2 - 2\epsilon_{k+1}(2M + \Lambda(\bar{q} + 1)).
\end{aligned} \tag{7.66}$$

By (7.48), (7.52), (7.62), and (7.64),

$$\begin{aligned}
&| \|y_{i-1}^{(k,T)} - y_i^{(k,T)}\|^2 - \|y_{i-1}^{(k,T)} - T_i(y_{i-1}^{(k,T)})\|^2 | \\
&\le | \|y_{i-1}^{(k,T)} - y_i^{(k,T)}\| - \|y_{i-1}^{(k,T)} - T_i(y_{i-1}^{(k,T)})\| | \\
&\times (\|y_{i-1}^{(k,T)} - y_i^{(k,T)}\| + \|y_{i-1}^{(k,T)} - T_i(y_{i-1}^{(k,T)})\|) \\
&\le \|y_{i-1}^{(k,T)} - T_i(y_{i-1}^{(k,T)})\| \\
&\times (\|y_{i-1}^{(k,T)} - z\| + \|z - y_i^{(k,T)}\| + \|y_{i-1}^{(k,T)} - z\| + \|z - T_i(y_{i-1}^{(k,T)}\|) \\
&\le 4\epsilon_{k+1}(2M + \Lambda(\bar{q} + 1)).
\end{aligned} \tag{7.67}$$

In view of (7.66) and (7.67),

$$\|z - y_{i-1}^{(k,T)}\|^2 - \|z - y_i^{(k,T)}\|^2$$
$$\geq \bar{c}\|y_{i-1}^{(k,T)} - y_i^{(k,T)}\|^2 - 6\epsilon_{k+1}(2M + \Lambda(\bar{q} + 1)). \tag{7.68}$$

It follows from (7.22), (7.47), (7.49), (7.50), and (7.68) that

$$\|z - x_k\|^2 - \|z - y_{k,T}\|^2$$
$$= \|z - y_0^{(k,T)}\|^2 - \|z - y_{p(T)}^{(k,T)}\|^2$$
$$= \sum_{i=1}^{p(T)} [\|z - y_{i-1}^{(k,T)}\|^2 - \|z - y_i^{(k,T)}\|^2]$$
$$\geq \sum_{i=1}^{p(T)} [\bar{c}\|y_{i-1}^{(k,T)} - y_i^{(k,T)}\|^2 - 6\epsilon_{k+1}(2M + \Lambda(\bar{q} + 1))]$$
$$\geq \bar{c} \sum_{i=1}^{p(T)} \|y_{i-1}^{(k,T)} - y_i^{(k,T)}\|^2 - 6\epsilon_{k+1}\bar{q}(2M + \Lambda(\bar{q} + 1))$$
$$\geq \bar{c}\lambda_{k,T}^2 - 6\epsilon_{k+1}\bar{q}(2M + \Lambda(\bar{q} + 1)). \tag{7.69}$$

By (7.16), (7.23), (7.46), (7.69), and the convexity of the function $\| \cdot \|^2$,

$$\|z - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\|^2$$
$$\leq \sum_{T \in \Omega_{k+1}} w_{k+1}(T)\|z - y_{k,T}\|^2$$
$$\leq \sum_{T \in \Omega_{k+1}} w_{k+1}(T)(\|z - x_k\|^2 - \bar{c}\lambda_{k,T}^2 + 6\epsilon_{k+1}\bar{q}(2M + \Lambda(\bar{q} + 1)))$$
$$\leq \|z - x_k\|^2 - \bar{c} \sum_{T \in \Omega_{k+1}} w_{k+1}(T)\lambda_{k,T}^2 + 6\epsilon_{k+1}\bar{q}(2M + \Lambda(\bar{q} + 1))$$
$$\leq \|z - x_k\|^2 - \bar{c}\Delta\lambda_{k+1}^2 + 6\epsilon_{k+1}\bar{q}(2M + \Lambda(\bar{q} + 1)). \tag{7.70}$$

By (7.16), (7.45), (7.62), (7.63), and the convexity of the function $\| \cdot \|$,

$$|\|z - x_{k+1}\|^2 - \|z - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\|^2|$$

$$\leq |\|z - x_{k+1}\| - \|z - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\||$$
$$\times(\|z - x_{k+1}\| + \|z - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\|)$$
$$\leq \|x_{k+1} - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\|$$
$$\times(\|z - x_{k+1}\| + \sum_{T \in \Omega_{k+1}} w_{k+1}(T) \|z - y_{k,T}\|)$$
$$\leq 2\epsilon_{k+1}(2M + \Lambda(\bar{q} + 1)). \tag{7.71}$$

Relations (7.70) and (7.71) imply that

$$\|z - x_{k+1}\|^2$$
$$\leq \|z - x_k\|^2 - \bar{c}\Delta\lambda_{k+1}^2 + 8\epsilon_{k+1}\bar{q}(2M + \Lambda(\bar{q} + 1)). \tag{7.72}$$

Let $n > n_0$ be an integer. In view of (7.62) and (7.72),

$$(2M + \Lambda(\bar{q} + 1))^2 \geq \|z - x_{n_0}\|^2$$
$$\geq \|z - x_{n_0}\|^2 - \|z - x_n\|^2$$
$$= \sum_{k=n_0}^{n-1} [\|z - x_k\|^2 - \|z - x_{k+1}\|^2]$$
$$\geq \sum_{k=n_0}^{n-1} [\bar{c}\Delta\lambda_{k+1}^2 - 8\bar{q}(2M + \Lambda(\bar{q} + 1))\epsilon_{k+1}].$$

Together with (7.27) this implies that

$$(2M + \Lambda(\bar{q} + 1))^2 + 8\bar{q}(2M + \Lambda(\bar{q} + 1))\Lambda$$
$$\geq \bar{c}\Delta \sum_{k=n_0}^{n-1} \lambda_{k+1}^2$$
$$\geq \bar{c}\Delta\gamma_0^2 \mathrm{Card}(\{k \in \{n_0, \dots, n-1\} : \lambda_{k+1} \geq \gamma_0\}).$$

Since the relation above holds for any natural number $n > n_0$ we conclude that

$$\mathrm{Card}(\{k \in \{n_0, n_0 + 1, \dots\} : \lambda_{k+1} \geq \gamma_0\})$$
$$\leq \bar{c}^{-1}\Delta^{-1}\gamma_0^{-2}[(2M + \Lambda(\bar{q} + 1))^2 + 8\bar{q}(2M + \Lambda(\bar{q} + 1))\Lambda]. \tag{7.73}$$

Set

$$E = \{k \in \{n_0, n_0 + 1, \dots\} :$$
$$\max\{\lambda_{i+1} : i \in \{k, \dots, k + \bar{N} - 1\}\} \geq \gamma_0\}. \tag{7.74}$$

By (7.73) and (7.74),

$$\mathrm{Card}(E)$$
$$\leq \bar{N}\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2}[(2M + \Lambda(\bar{q} + 1))^2 + 8\bar{q}\Lambda(2M + \Lambda(\bar{q} + 1))]. \tag{7.75}$$

In view of (7.28) and (7.41), for all integers $k \geq n_0$,

$$\epsilon_{k+1} < \gamma_0. \tag{7.76}$$

Assume that an integer q satisfies

$$q \geq n_0, \ q \notin E. \tag{7.77}$$

In view of (7.74) and (7.77),

$$\lambda_{k+1} < \gamma_0 \text{ for all } k \in \{q, \dots, q + \bar{N} - 1\}. \tag{7.78}$$

It follows from (7.16), (7.22), (7.45)–(7.50), (7.76)–(7.78), and the convexity of the norm that for each $k \in \{q, \dots, q + \bar{N} - 1\}$, each $T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}$, and each $j \in \{1, \dots, p(T)\}$,

$$\gamma_0 > \lambda_{k+1} \geq \lambda_{k,T} \geq \|y_j^{(k,T)} - y_{j-1}^{(k,T)}\|, \tag{7.79}$$

$$\|x_k - y_j^{(k,T)}\|, \ \|x_k - y_{j-1}^{(k,T)}\| \leq \gamma_0 j \leq \bar{q}\gamma_0, \tag{7.80}$$

$$\|x_k - y_{k,T}\| \leq \bar{q}\gamma_0, \tag{7.81}$$

$$\|y_{j-1}^{(k,T)} - T_j(y_{j-1}^{(k,T)})\|$$
$$\leq \|y_{j-1}^{(k,T)} - y_j^{(k,T)}\| + \|y_j^{(k,T)} - T_j(y_{j-1}^{(k,T)})\|$$
$$\leq \gamma_0 + \epsilon_{k+1} \leq 2\gamma_0, \tag{7.82}$$

$$\|x_k - x_{k+1}\|$$
$$\leq \|x_k - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\| + \|\sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T} - x_{k+1}\|$$
$$\leq \sum_{T \in \Omega_{k+1}} w_{k+1}(T)\|x_k - y_{k,T}\| + \epsilon_{k+1}$$

$$\leq \bar{q}\gamma_0 + \epsilon_{k+1} < (\bar{q}+1)\gamma_0. \tag{7.83}$$

Relation (7.83) implies that for each $k_1, k_2 \in \{q, \dots, q+\bar{N}\}$,

$$\|x_{k_1} - x_{k_2}\| \leq \gamma_0 \bar{N}(\bar{q}+1). \tag{7.84}$$

Let

$$Q \in \mathcal{L}_2. \tag{7.85}$$

Property (P1) and (7.85) imply that there exist

$$k \in \{q, \dots, q+\bar{N}-1\},\ T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1},\ s \in \{1, \dots, p(T)\} \tag{7.86}$$

such that

$$Q = T_s. \tag{7.87}$$

In view of (7.10), (7.82), and (7.87),

$$y_{s-1}^{(k,T)} \in \mathrm{Fix}_{2\gamma_0}(Q). \tag{7.88}$$

By (7.41), (7.80), and (7.88),

$$d(x_k, \mathrm{Fix}_{2\gamma_0}(Q)) \leq \|x_k - y_{s-1}^{(k,T)}\| \leq \gamma_0 \bar{q}. \tag{7.89}$$

It follows from (7.41), (7.84), and (7.89) that

$$d(x_q, \mathrm{Fix}_{2\gamma_0}(Q)) \leq \|x_q - x_k\| + d(x_k, \mathrm{Fix}_{2\gamma_0}(Q)) \leq \bar{N}(\bar{q}+1)\gamma_0 + \bar{q}\gamma_0$$
$$\leq (\bar{q}+1)(\bar{N}+1)\gamma_0 \leq \epsilon$$

and

$$d(x_q, \mathrm{Fix}_{\epsilon}(Q)) \leq \epsilon \text{ for all } Q \in \mathcal{L}_2. \tag{7.90}$$

Let

$$Q \in \mathcal{L}_1.$$

Property (P2) and the inclusion above imply that there exist

$$k \in \{q, \dots, q+\bar{N}-1\},\ T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1},$$
$$s \in \{1, \dots, p(T)\},\ c \geq \bar{\lambda} \tag{7.91}$$

such that

$$P_{c,Q} = T_s. \tag{7.92}$$

By (7.82) and (7.92),

$$\begin{aligned} 2\gamma_0 &> \|y_{s-1}^{(k,T)} - T_s(y_{s-1}^{(k,T)})\| \\ &= \|y_{s-1}^{(k,T)} - P_{c,Q}(y_{s-1}^{(k,T)})\|. \end{aligned} \tag{7.93}$$

By (7.1) and (7.93),

$$\begin{aligned} y_{s-1}^{(k,T)} &\in (I + cQ)(P_{c,Q}(y_{s-1}^{(k,T)})), \\ y_{s-1}^{(k,T)} - P_{c,Q}(y_{s-1}^{(k,T)}) &\in cQ(P_{c,Q}(y_{s-1}^{(k,T)})), \\ c^{-1}(y_{s-1}^{(k,T)} - P_{c,Q}(y_{s-1}^{(k,T)})) &\in Q(P_{c,Q}(y_{s-1}^{(k,T)})). \end{aligned} \tag{7.94}$$

It follows from (7.91) and (7.93) that

$$\|c^{-1}(y_{s-1}^{(k,T)} - P_{c,Q}(y_{s-1}^{(k,T)}))\| \le 2\bar{\lambda}^{-1}\gamma_0. \tag{7.95}$$

By (7.94) and (7.95),

$$P_{c,Q}(y_{s-1}^{(k,T)}) \in F_{2\bar{\lambda}^{-1}\gamma_0}(Q). \tag{7.96}$$

In view of (7.48), (7.76)–(7.78), (7.91), and (7.92),

$$\|y_s^{(k,T)} - P_{c,Q}(y_{s-1}^{(k,T)})\| \le \epsilon_{k+1} < \gamma_0. \tag{7.97}$$

By (7.96) and (7.97),

$$d(y_s^{(k,T)}, F_{2\bar{\lambda}^{-1}\gamma_0}(Q)) < \gamma_0. \tag{7.98}$$

Relations (7.80) and (7.98) imply that

$$d(x_k, F_{2\bar{\lambda}^{-1}\gamma_0}(Q)) \le \|x_k - y_s^{(k,T)}\| + d(y_s^{(k,T)}, F_{2\bar{\lambda}^{-1}\gamma_0}(Q)) \le \bar{q}\gamma_0 + \gamma_0.$$

It follows from (7.41), (7.84), (7.92), and the inequality above that

$$\begin{aligned} d(x_q, F_\epsilon(Q)) &\le \|x_q - x_k\| + d(x_k, F_\epsilon(Q)) \\ &\le (\bar{q} + 1)\gamma_0\bar{N} + (\bar{q} + 1)\gamma_0 \le \epsilon \end{aligned}$$

and

$$d(x_q, F_\epsilon(Q)) \le \epsilon \text{ for all } Q \in \mathcal{L}_1.$$

Together with (7.90) this implies that

$$x_q \in \tilde{F}_\epsilon$$

for all integers $q \geq n_0$ satisfying $q \notin E$. Together with (7.41) and (7.75) this implies that

$$\begin{aligned}
&\mathrm{Card}(\{i \in \{0, 1, \dots\} : \ x_i \notin \tilde{F}_\epsilon\}) \\
&\leq n_0 + \mathrm{Card}(E) \\
&\leq n_0 + \bar{N}\bar{c}^{-1}\Delta^{-1}[(2M + \Lambda(\bar{q}+1))^2 \\
&+8\bar{q}\Lambda(2M + (\bar{q}+1)\Lambda]\epsilon^{-2}(\bar{N}+1)(\bar{q}+1)^2(\min\{1, \bar{\lambda}\})^{-2}.
\end{aligned}$$

Theorem 7.1 is proved.

7.3 Proof of Theorem 7.2

Set

$$\gamma_0 = \epsilon(\bar{N}+1)^{-2}(\bar{q}+1)^{-2}\min\{1, \bar{\lambda}\}\bar{c}\Delta(4M)^{-1}. \tag{7.99}$$

By (7.33) there exists

$$z \in B(0, M) \cap F. \tag{7.100}$$

Let $k \geq 0$ be an integer. By (7.37),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, (\Omega_{k+1}, w_{k+1}), 0). \tag{7.101}$$

By (7.25) and (7.101) there exist

$$(y_{k,T}, \lambda_{k,T}) \in A_0(x_k, T, 0), \ \ T \in \Omega_{k+1} \tag{7.102}$$

such that

$$x_{k+1} = \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}, \tag{7.103}$$

$$\lambda_{k+1} = \max\{\lambda_{k,T} : \ T \in \Omega_{k+1}\}. \tag{7.104}$$

Let

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}.$$

It follows from (7.24) and (7.103) that there exists a finite sequence

$$\{y_i^{(k,T)}\}_{i=0}^{p(T)} \subset X$$

such that

$$y_0^{(k,T)} = x_k, \tag{7.105}$$

$$y_i^{(k,T)} = T_i(y_{i-1}^{(k,T)}) \text{ for each integer } i = 1, \dots, p(T), \tag{7.106}$$

$$y_{p(T)}^{(k,T)} = y_{k,T}, \tag{7.107}$$

$$\lambda_{k,T} = \max\{\|y_i^{(k,T)} - y_{i-1}^{(k,T)}\| : \ i = 1, \dots, p(T)\}. \tag{7.108}$$

By (7.7), (7.12), (7.34), (7.100), (7.106), and Lemma 6.7, for each

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}$$

and each $i \in \{1, \dots, p(T)\}$, we have

$$\begin{aligned} \|z - y_{i-1}^{(k,T)}\|^2 &\geq \|z - T_i(y_{i-1}^{(k,T)})\|^2 + \bar{c}\|y_{i-1}^{(k,T)} - T_i(y_{i-1}^{(k,T)})\|^2 \\ &= \|z - y_i^{(k,T)}\|^2 + \bar{c}\|y_{i-1}^{(k,T)} - y_i^{(k,T)}\|^2, \end{aligned} \tag{7.109}$$

and

$$\|z - y_i^{(k,T)}\| \leq \|z - y_{i-1}^{(k,T)}\|. \tag{7.110}$$

Relations (7.105), (7.107), and (7.110) imply that for each

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}$$

and each $i \in \{0, 1, \dots, p(T)\}$, we have

$$\|z - y_i^{(k,T)}\| \leq \|z - y_0^{(k,T)}\| = \|z - x_k\|, \tag{7.111}$$

$$\|z - y_{k,T}\| = \|z - y_{p(T)}^{(k,T)}\| \leq \|z - x_k\|. \tag{7.112}$$

It follows from (7.13), (7.17), (7.103), and (7.105)–(7.107) that

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k). \tag{7.113}$$

Set

$$Q = \prod_{i=1}^{\bar{N}} P_{\Omega_i, w_i} = P_{\Omega_{\bar{N}}, w_{\bar{N}}} \cdots P_{\Omega_1, w_1}. \tag{7.114}$$

In view of (7.35), (7.113), and (7.114), for each integer $k \geq 0$,

$$x_{(k+1)\bar{N}} = \prod_{i=k\bar{N}+1}^{(k+1)\bar{N}} P_{\Omega_i, w_i}(x_{k\bar{N}}) = Q(x_{k\bar{N}}). \tag{7.115}$$

By (7.2), (7.16), (7.17), (7.32), (7.72), and (7.114), for each $x, y \in X$,

$$\|Q(x) - Q(y)\| \leq \|x - y\|. \tag{7.116}$$

It follows from (7.16), (7.103), (7.112) and the convexity of the norm that for each integer $k \geq 0$,

$$\begin{aligned}
&\|z - x_{k+1}\| \\
&= \|z - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\| \\
&\leq \sum_{T \in \Omega_{k+1}} w_{k+1}(T) \|z - y_{k,T}\| \\
&\leq \|z - x_k\|.
\end{aligned} \tag{7.117}$$

In view (7.36), (7.100), and (7.117),

$$\|z - x_k\| \leq \|z - x_0\| \leq 2M. \tag{7.118}$$

By (7.105) and (7.107)–(7.109), for each integer $k \geq 0$ and for each

$$T = (T_1, \ldots, T_{p(T)}) \in \Omega_{k+1},$$

we have

$$\begin{aligned}
&\|z - x_k\|^2 - \|z - y_{k,T}\|^2 \\
&= \|z - y_0^{(k,T)}\|^2 - \|z - y_{p(T)}^{(k,T)}\|^2 \\
&= \sum_{i=1}^{p(T)} [\|z - y_{i-1}^{(k,T)}\|^2 - \|z - y_i^{(k,T)}\|^2] \\
&\geq \sum_{i=1}^{p(T)} \bar{c} \|y_{i-1}^{(k,T)} - y_i^{(k,T)}\|^2 \geq \bar{c} \lambda_{k,T}^2.
\end{aligned} \tag{7.119}$$

By (7.16), (7.23), (7.103), (7.104), (7.119), and the convexity of the function $\|\cdot\|^2$, for each integer $k \geq 0$,

$$\|z - x_{k+1}\|^2$$
$$= \|z - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\|^2$$
$$\leq \sum_{T \in \Omega_{k+1}} w_{k+1}(T)\|z - y_{k,T}\|^2$$
$$\leq \sum_{T \in \Omega_{k+1}} w_{k+1}(T)(\|z - x_k\|^2 - \bar{c}\lambda_{k,T}^2)$$
$$\leq \|z - x_k\|^2 - \bar{c}\Delta\lambda_{k+1}^2$$

and

$$\|z - x_{k+1}\|^2 \leq \|z - x_k\|^2 - \bar{c}\Delta\lambda_{k+1}^2. \tag{7.120}$$

Let n be a natural number. In view of (7.118) and (7.120),

$$4M^2 \geq \|z - x_0\|^2$$
$$\geq \|z - x_0\|^2 - \|z - x_{\bar{N}n}\|^2$$
$$= \sum_{k=0}^{n-1}[\|z - x_{k\bar{N}}\|^2 - \|z - x_{(k+1)\bar{N}}\|^2]$$
$$= \sum_{k=0}^{n-1}[\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1}(\|z - x_j\|^2 - \|z - x_{j+1}\|^2)]$$
$$\geq \sum_{k=0}^{n-1}[\sum_{j=k\bar{N}}^{(k+1)\bar{N}-1}\bar{c}\Delta\lambda_{i+1}^2]$$
$$\geq \bar{c}\Delta\gamma_0^2 \mathrm{Card}(\{k \in \{0, \dots, n-1\} :$$
$$\max\{\lambda_{j+1} : \ j = k\bar{N}, \dots, (k+1)\bar{N} - 1\} \geq \gamma_0\})$$

and

$$\mathrm{Card}(\{k \in \{0, \dots, n-1\} :$$
$$\max\{\lambda_{j+1} : \ j = k\bar{N}, \dots, (k+1)\bar{N} - 1\} \geq \gamma_0\})$$
$$\leq 4M^2\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2}.$$

Since the relation above holds for any natural number n we conclude that

$$\begin{aligned}&\mathrm{Card}(\{k \in \{0, 1, \dots\} : \\ &\max\{\lambda_{j+1} : \ j = k\bar{N}, \dots, (k+1)\bar{N} - 1\} \ge \gamma_0\}) \\ &\le 4M^2\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2}. \end{aligned} \tag{7.121}$$

In view of (7.121), there exists an integer

$$q_0 \le 4\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2}M^2 + 1 \tag{7.122}$$

such that

$$\lambda_{j+1} < \gamma_0, \ \ j = q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1. \tag{7.123}$$

It follows from (7.16), (7.22), (7.103)–(7.105), (7.107), (7.108), (7.123), and the convexity of the function $\|\cdot\|$ that for all $k = q_0\bar{N}, \dots, (q_0+1)\bar{N} - 1$,

$$\begin{aligned}&\|x_k - x_{k+1}\| \\ &\le \|x_k - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\| \\ &\le \sum_{T \in \Omega_{k+1}} w_{k+1}(T)\|x_k - y_{k,T}\| \\ &\le \sum_{T \in \Omega_{k+1}} (\lambda_{k+1} p(T)) w_{k+1}(T) \le \lambda_{k+1}\bar{q} \le \bar{q}\gamma_0. \end{aligned} \tag{7.124}$$

In view of (7.115) and (7.124),

$$\|x_{q_0\bar{N}} - Q(x_{q_0\bar{N}})\| = \|x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}}\| \le \bar{q}\gamma_0\bar{N}. \tag{7.125}$$

By (7.115), (7.116), and (7.125), for each integer $q > q_0$,

$$\begin{aligned}\|x_{q\bar{N}} - x_{(q+1)\bar{N}}\| &= \|Q^{q-q_0}(x_{q_0\bar{N}}) - Q^{q-q_0}(x_{(q_0+1)\bar{N}})\| \\ &\le \|x_{q_0\bar{N}} - x_{(q_0+1)\bar{N}}\| \le \bar{q}\gamma_0\bar{N}. \end{aligned}$$

Together with (7.125) this implies that

$$\|x_{q\bar{N}} - x_{(q+1)\bar{N}}\| \le \bar{q}\gamma_0\bar{N} \text{ for all integers } q \ge q_0. \tag{7.126}$$

Let $q \ge q_0$ be an integer. In view of (7.118), (7.120), and (7.126),

$$\begin{aligned}\bar{q}\gamma_0\bar{N} &\ge \|x_{q\bar{N}} - x_{(q+1)\bar{N}}\| \\ &\ge \|z - x_{q\bar{N}}\| - \|z - x_{(q+1)\bar{N}}\| \\ &\ge (\|z - x_{q\bar{N}}\|^2 - \|z - x_{(q+1)\bar{N}}\|^2)(4M)^{-1} \\ &= (4M)^{-1}(\sum_{k=q\bar{N}}^{(q+1)\bar{N}-1} (\|z - x_k\|^2 - \|z - x_{k+1}\|^2) \\ &\ge (4M)^{-1} \sum_{k=q\bar{N}}^{(q+1)\bar{N}-1} \bar{c}\Delta\lambda_{k+1}^2.\end{aligned}$$

This implies that for all $k = q\bar{N}, \dots, (q+1)\bar{N} - 1$,

$$\lambda_{k+1} \le 4\bar{q}\gamma_0\bar{N}M\bar{c}^{-1}\Delta^{-1}. \tag{7.127}$$

By (7.16), (7.22), (7.103)–(7.105), (7.107), (7.108), (7.127), and the convexity of the norm, for each integer $k \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$, each

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}$$

and each $j \in \{1, \dots, p(T)\}$, we have

$$4\bar{c}^{-1}\Delta^{-1}\bar{q}\gamma_0\bar{N}M \ge \lambda_{k+1} \ge \lambda_{k,T} \ge \|y_j^{(k,T)} - y_{j-1}^{(k,T)}\|, \tag{7.128}$$

$$\begin{aligned}&\|x_k - y_j^{(k,t)}\|,\ \|x_k - y_{j-1}^{(k,t)}\| \\ &\le 4j\bar{c}^{-1}\Delta^{-1}\bar{q}\gamma_0\bar{N}M \le 4\bar{c}^{-1}\Delta^{-1}\bar{q}^2\gamma_0\bar{N}M,\end{aligned} \tag{7.129}$$

$$\|x_k - y_{k,T}\| \le 4\bar{c}^{-1}\Delta^{-1}\bar{q}^2\gamma_0\bar{N}M, \tag{7.130}$$

$$\begin{aligned}\|x_k - x_{k+1}\| &= \|x_k - \sum_{T\in\Omega_{k+1}} w_k(T)y_{k,T}\| \\ &\le \sum_{T\in\Omega_{k+1}} w_{k1}(T)\|x_k - y_{k,T}\| \\ &\le 4\bar{c}^{-1}\Delta^{-1}\bar{q}^2\gamma_0\bar{N}M.\end{aligned} \tag{7.131}$$

In view of (7.131), for each $k_1, k_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$\|x_{k_1} - x_{k_2}\| \le 4\bar{c}^{-1}\Delta^{-1}\bar{q}^2\gamma_0\bar{N}^2M. \tag{7.132}$$

Set

$$\gamma_1 = 4\bar{c}^{-1}\Delta^{-1}\bar{q}\gamma_0\bar{N}M. \tag{7.133}$$

Let

$$S \in \mathcal{L}_2. \tag{7.134}$$

Property (P1) and (7.134) imply that there exist

$$k \in \{q, \dots, (q+1)\bar{N} - 1\},\ T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1},\ i \in \{1, \dots, p(T)\} \tag{7.135}$$

such that

$$S = T_i. \tag{7.136}$$

In view of (7.37), (7.106), (7.128), and (7.132),

$$\gamma_1 \geq \|y_{i-1}^{(k,T)} - y_i^{(k,T)}\| = \|y_{i-1}^{(k,T)} - T_i(y_{i-1}^{(k,T)})\| = \|y_{i-1}^{(k,T)} - S(y_{i-1}^{(k,T)})\|$$

and

$$y_{i-1}^{(k,T)} \in \mathrm{Fix}_{\gamma_1}(S). \tag{7.137}$$

By (7.22), (7.129), (7.133), and (7.137),

$$\begin{aligned} d(x_k, \mathrm{Fix}_{\gamma_1}(S)) &\leq \|x_k - y_{i-1}^{(k,T)}\| \\ &\leq 4\bar{c}^{-1}\Delta^{-1}\bar{q}^2\gamma_0\bar{N}M = \bar{q}\gamma_1. \end{aligned} \tag{7.138}$$

It follows from (7.99), (7.132)–(7.135), and (7.138) that for all

$$p \in \{q\bar{N}, \dots, (q+1)\bar{N}\},$$

we have

$$\begin{aligned} d(x_p, \mathrm{Fix}_{\gamma_1}(S)) &\leq \|x_p - x_k\| + d(x_k, \mathrm{Fix}_{\gamma_1}(S)) \\ &\leq \bar{N}\bar{q}\gamma_1 + \bar{q}\gamma_1 \\ &\leq \bar{q}(\bar{N}+1)\gamma_1 \leq \epsilon \end{aligned}$$

and

$$d(x_p, \mathrm{Fix}_{\gamma_1}(S)) \leq \epsilon \text{ for all } S \in \mathcal{L}_2. \tag{7.139}$$

Let

$$S \in \mathcal{L}_1. \tag{7.140}$$

Property (P2) and (7.140) imply that there exist

$$k \in \{q, \dots, q + \bar{N} - 1\},\ T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1},$$
$$i \in \{1, \dots, p(T)\},\ c \geq \bar{\lambda} \tag{7.141}$$

such that

$$P_{c,S} = T_i. \tag{7.142}$$

By (7.106), (7.119), (7.133), and (7.142),

$$\gamma_1 \geq \|y_{i-1}^{(k,T)} - y_i^{(k,T)}\|$$
$$= \|y_{i-1}^{(k,T)} - T_i(y_{i-1}^{(k,T)})\| = \|y_{i-1}^{(k,T)} - P_{c,S}(y_{i-1}^{(k,T)})\|. \tag{7.143}$$

By (7.1) and (7.143),

$$y_{i-1}^{(k,T)} \in (I + cS)(P_{c,S}(y_{i-1}^{(k,T)})),$$
$$y_{i-1}^{(k,T)} - P_{c,S}(y_{i-1}^{(k,T)}) \in cS(P_{c,S}(y_{i-1}^{(k,T)})),$$
$$c^{-1}(y_{i-1}^{(k,T)} - P_{c,S}(y_{i-1}^{(k,T)})) \in S(P_{c,S}(y_{i-1}^{(k,T)})). \tag{7.144}$$

It follows from (7.141) and (7.143) that

$$\|c^{-1}(y_{i-1}^{(k,T)} - P_{c,S}(y_{i-1}^{(k,T)}))\| \leq \bar{\lambda}^{-1}\gamma_1. \tag{7.145}$$

By (7.106), (7.142), (7.144), and (7.145),

$$y_i^{(k,T)} = T_i(y_{i-1}^{(k,T)}) = P_{c,S}(y_{i-1}^{(k,T)}) \in F_{\bar{\lambda}^{-1}\gamma_1}(S). \tag{7.146}$$

In view of (7.129), (7.133), and (7.146),

$$d(x_k, F_{\bar{\lambda}^{-1}\gamma_1}(S)) \leq \|x_k - y_i^{(k,T)}\|$$
$$\leq 4\bar{c}^{-1}\Delta^{-1}\bar{q}^2\gamma_0\bar{N}M = \gamma_1\bar{q}. \tag{7.147}$$

By (7.99), (7.132), (7.133), (7.141), and (7.147), for all $p \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$d(x_p, F_\epsilon(S)) \leq d(x_p, F_{\bar{\lambda}^{-1}\gamma_1}(S))$$

$$\leq \|x_p - x_k\| + d(x_k, F_{\bar{\lambda}^{-1}\gamma_1}(S)))$$
$$\leq \gamma_1 \bar{q}\bar{N} + \gamma_1 \bar{q} \leq \gamma_1 \bar{q}(\bar{N}+1) \leq \epsilon$$

and

$$d(x_p, F_\epsilon(S)) \leq \epsilon \text{ for all } S \in \mathcal{L}_1.$$

Together with (7.140) this implies that

$$x_q \in \tilde{F}_\epsilon$$

for all integers $q \geq q_0$ and all integers $p \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$. This completes the proof of Theorem 7.2.

7.4 Proof of Theorem 7.3

Theorem 7.3 is deduced from Theorems 2.9 and 7.2. Let $Y = X$, $N = \bar{N}$, $\rho(y,z) = \|y - z\|$, $y, z \in X$, $\mathfrak{A}$ be the set of all mappings S defined on the set of natural numbers into the set of all nonexpansive self-mappings of X for which there exists

$$\{(\Omega_i^{(S)}, w_i^{(S)})\}_{i=1}^\infty \in \mathcal{R}$$

such that

$$P_{\Omega_i^{(S)}, w_i^{(S)}} = P_{\Omega_{i+\bar{N}}^{(S)}, w_{i+\bar{N}}^{(S)}} \text{ for all integers } i \geq 0$$

and

$$S(i) = P_{\Omega_i^{(S)}, w_i^{(S)}} \text{ for all integers } i \geq 1.$$

Set

$$F = \tilde{F}_{\epsilon_0/4}.$$

Theorem 7.2 implies that for every $M > 0$ property (P6) holds with

$$Q = \lfloor \bar{N}(1 + 64M^4\bar{c}^{-3}\Delta^{-3}\epsilon_0^{-2}(\bar{q}+1)^4(\bar{N}+1)^4 \min\{1, \bar{\lambda}\}^{-2}) \rfloor.$$

By (7.25), for each integer $k \geq 0$, there exist vectors

$$(y_{k,T}, \lambda_{k,T}) \in A_0(x_k, T, \delta),\ \ T \in \Omega_{k+1} \tag{7.148}$$

such that

$$\|x_{k+1} - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\| \leq \delta. \tag{7.149}$$

Proposition 2.8, (7.22), (7.29), and (7.148) imply that for every integer $k \geq 0$ and every $T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}$,

$$\|y_{k,T} - P[t](x_k)\| \leq \bar{q}\delta. \tag{7.150}$$

By (7.16), (7.149), and (7.150), for each integer $k \geq 0$,

$$\begin{aligned} &\|x_{k+1} - P_{\Omega_k, w_k}(x_k)\| \\ &\leq \|x_{k+1} - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\| \\ &+\|\sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T} - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) P[T](x_k)\| \\ &\leq (\bar{q} + 1)\delta. \end{aligned} \tag{7.151}$$

Theorems 2.9 and 7.2 and (7.151) imply that for all integer $i \geq Q_0$,

$$B(x_i, \epsilon_0/4) \cap F \neq \emptyset.$$

By the inclusion above,

$$x_i \in \tilde{F}_{\epsilon_0}$$

for all integers $i \geq Q_0$. Theorem 7.3 is proved.

Chapter 8
Convex Feasibility Problems

We use inexact subgradient projection algorithms for solving convex feasibility problems. We show that almost all iterates, generated by a perturbed subgradient projection algorithm in a Hilbert space, are approximate solutions. Moreover, we obtain an estimate of the number of iterates which are not approximate solutions.

8.1 Preliminaries

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, which induces a complete norm $\| \cdot \|$. For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) := \inf\{\|x - y\| : \ y \in A\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

It is well known that the following proposition holds (see Fact 1.5 and Lemma 2.4 of [8]).

Proposition 8.1 *Let C be a nonempty, closed and convex subset of X. Then, for each $x \in X$, there is a unique point $P_C(x) \in C$ satisfying*

$$\|x - P_C(x)\| = d(x, C).$$

Moreover, $\|P_C(x) - P_C(y)\| \le \|x - y\|$ for all $x, y \in X$ and, for each $x \in X$ and each $z \in C$,

$$\langle z - P_C(x), x - P_C(x) \rangle \le 0,$$

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$$\|z - P_C(x)\|^2 + \|x - P_C(x)\|^2 \le \|z - x\|^2. \tag{8.1}$$

Let $f : X \to R^1$ be a continuous and convex function such that

$$\{x \in X : f(x) \le 0\} \ne \emptyset. \tag{8.2}$$

Let $y_0 \in X$. Then the set

$$\partial f(y_0) := \{l \in X : f(y) - f(y_0) \ge \langle l, y - y_0 \rangle \text{ for all } y \in X\} \tag{8.3}$$

is the subdifferential of f at the point y_0 [63, 94, 99]. For any $l \in \partial f(y_0)$, in view of (10.3),

$$\{x \in X : f(x) \le 0\} \subset \{x \in X : f(y_0) + \langle l, x - y_0 \rangle \le 0\}. \tag{8.4}$$

It is well known that the following lemma holds (see Lemma 7.3 of [8]).

Lemma 8.2 *Let $y_0 \in X$, $f(y_0) > 0$, $l \in \partial f(y_0)$ and let*

$$D := \{x \in X : f(y_0) + \langle l, x - y_0 \rangle \le 0\}.$$

Then $l \ne 0$ and $P_D(y_0) = y_0 - f(y_0)\|l\|^{-2} l$.

8.2 Iterative Methods

Let m be a natural number, $\mathbb{I} = \{1, \dots, m\}$ and $f_i : X \to R^1$, $i = 1, \dots, m$, be convex and continuous functions. For each $i \in \{1, \dots, m\}$ set

$$C_i := \{x \in X : f_i(x) \le 0\},$$
$$C := \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : f_i(x) \le 0\}.$$

Suppose that

$$C \ne \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given $\epsilon > 0$, a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if $f_i(x) \le \epsilon$ for all $i = 1, \dots, m$. We apply the subgradient projection method in order to obtain a good approximative solution of the feasibility problem.

Consider a natural number $\bar{p} \ge m$. Denote by $\mathcal{R}$ the set of all mappings S : $0, 1, \dots \to \{1, \dots, m\}$ such that the following property holds:

(P1) For each integer $N \geq 0$ and each $i \in \{1, \ldots, m\}$, there is $n \in \{N, \ldots, N + \bar{p} - 1\}$ such that $S(n) = i$.

We want to find approximate solutions of the inclusion $x \in C$. In order to meet this goal we apply algorithms generated by $S \in \mathcal{R}$.

For each $x \in X$, each number $\epsilon \geq 0$, and each $i \in \{1, \ldots, m\}$ set

$$A_i(x, \epsilon) := \{x\} \text{ if } f_i(x) \leq \epsilon \tag{8.5}$$

and, in view of Lemma 8.2,

$$A_i(x, \epsilon) := x - f_i(x)\{\|l\|^{-2}l : \ l \in \partial f_i(x)\} \text{ if } f_i(x) > \epsilon. \tag{8.6}$$

We associate with any $S \in \mathcal{R}$ the algorithm which generates, for any starting point $x_0 \in X$, a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ such that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, 0).$$

Note that by Lemma 8.2 the sequence $\{x_n\}_{n=0}^{\infty}$ is well defined, and that for each integer $n \geq 0$, if $f_{S(n)}(x_n) > 0$, then $x_{n+1} = P_{D_n}(x_n)$, where

$$D_n = \{x \in X : \ f(x_n) + \langle l_n, x - x_n \rangle \leq 0\} \text{ and } l_n \in \partial f_{S(n)}(x_n).$$

We will prove the following result (Theorem 8.3) which shows that, for the inexact subgradient projection method with summable errors, considered in the chapter, almost all iterates are good approximate solutions. Denote by $\mathrm{Card}(A)$ the cardinality of the set A.

Suppose that $M > 0$, $M_0 > 0$, $M_1 > 2$ and $\Lambda > 0$ be such that

$$B(0, M) \cap \{x \in X : \ f_i(x) \leq 0, \ i = 1, \ldots, m\} \neq \emptyset, \tag{8.7}$$

$$f_i(B(0, 3M + \Lambda)) \subset [-M_0, M_0], \ i = 1, \ldots, m, \tag{8.8}$$

$$|f_i(u) - f_i(v)| \leq (M_1 - 2)\|u - v\|$$

$$\text{for all } u, v \in B(0, 3M + \Lambda + 1) \text{ and all } i = 1, \ldots, m. \tag{8.9}$$

Theorem 8.3 *Let* $\Delta \in (0, 1]$,

$$\Delta_1 = 1 + 16M_0\Delta^{-2}(4M + \Lambda)^2, \tag{8.10}$$

$$\{\epsilon_i\}_{i=1}^{\infty} \subset (0, 2^{-1}\Delta(4M + \Lambda)^{-1}M_1^{-1}], \tag{8.11}$$

$$\sum_{i=1}^{\infty} \epsilon_i \leq \Delta_1^{-1}\Lambda, \tag{8.12}$$

$$\gamma_0 = 4^{-1}\Delta\bar{p}^{-1}M_1^{-1}. \tag{8.13}$$

Assume that

$$S \in \mathcal{R},\ \{x_k\}_{k=0}^{\infty} \subset X,\ \|x_0\| \le M,\ \{l_k\}_{k=0}^{\infty} \subset X, \tag{8.14}$$

and that for each integer $k \ge 0$,

$$x_{k+1} = x_k,\ l_k = 0,\ \textit{if } f_{S(k)}(x_k) \le \Delta, \tag{8.15}$$

$$\textit{if } f_{S(k)}(x_k) > \Delta,\ \textit{then}$$

$$l_k \in X \setminus \{0\},\ d(l_k, \partial f_{S(k)}(x_k)) < \epsilon_{k+1} \tag{8.16}$$

and

$$\|x_{k+1} - x_k + f_{S(k)}(x_k)\|l_k\|^{-2} l_k\| \le \epsilon_{k+1}. \tag{8.17}$$

Then

$$\|x_i\| \le 3M + \Lambda \textit{ for all integers } i \ge 0$$

and

$$Card(\{n \in 0, 1, \dots\} :\ \max\{\|x_{i+1} - x_i\| :\ i = n, \dots, n + \bar{p} - 1\} > \gamma_0\})$$
$$\le \bar{p}\gamma_0^{-2}(4M + 6\Lambda(2M + \Lambda)).$$

Moreover, if an integer $n \ge 0$ *satisfies*

$$\|x_{i+1} - x_i\| \le \gamma_0,\ i = n, \dots, n + \bar{p} - 1,$$

then, for all integers $k = n, \dots, n + \bar{p}$ *and each* $j = 1, \dots, m$, $f_j(x_k) < 2\Delta$.

8.3 An Auxiliary Result

Lemma 8.4 *Let* $\delta, \bar{\Delta} \in (0, 1]$ *satisfy*

$$\delta \le 2^{-1}\bar{\Delta}(4M + \Lambda)^{-1}, \tag{8.18}$$

an integer $j \in \{1, \dots, m\}$,

$$x \in B(0, 3M + \Lambda),\ \ f_j(x) > \bar{\Delta}, \tag{8.19}$$

$$z \in B(0, M) \cap C \tag{8.20}$$

and

$$\xi \in \partial f_j(x),\ l \in B(\xi, \delta). \tag{8.21}$$

Then $l \neq 0$, $\xi \neq 0$,

$$y := x - f_j(x)\|\xi\|^{-2}\xi \tag{8.22}$$

satisfy

$$\|y - z\| \leq \|z - x\|,$$
$$\|y - z\|^2 \leq \|z - x\|^2 - \|x - y\|^2 \tag{8.23}$$

and for each

$$u \in B(x - f_j(x)\|l\|^{-2}l, \delta) \tag{8.24}$$

the following inequalities hold:

$$\|u - y\| \leq \delta(1 + 16M_0\bar{\Delta}^{-2}(4M + \Lambda)^2), \tag{8.25}$$
$$\|u - z\| \leq \|x - z\| + \delta(1 + 16M_0\bar{\Delta}^{-2}(4M + \Lambda)^2). \tag{8.26}$$

Proof Define

$$D = \{v \in X : \ f_j(x) + \langle \xi, v - x\rangle \leq 0\}. \tag{8.27}$$

In view of (8.19) and (8.21), $\xi \neq 0$.

Lemma 8.2, (8.19), (8.21), and (8.23) imply that

$$P_D(x) = y. \tag{8.28}$$

By Proposition 8.1, (8.4), (8.20), (8.27), and (8.28),

$$\|z - y\|^2 = \|z - P_D(x)\|^2$$
$$\leq \|z - x\|^2 - \|x - y\|^2. \tag{8.29}$$

It is clear that (8.29) implies (8.23).

It follows from (8.8) and (8.19) that

$$|f_j(x)| \leq M_0. \tag{8.30}$$

Relations (8.9), (8.19), and (8.21) imply that

$$\|\xi\| \leq M_1 - 2,$$
$$\|l\| \leq M_1 - 1. \tag{8.31}$$

By (8.19)–(8.21),

$$-\bar{\Delta} \geq f_j(z) - f_j(x) \geq \langle \xi, z - x \rangle \geq -\|\xi\| \|z - x\|$$
$$\geq -\|\xi\|(4M + \Lambda)$$

and

$$\|\xi\| \geq \bar{\Delta}(4M + \Lambda)^{-1}. \tag{8.32}$$

Relations (8.18), (8.21), and (8.32) imply that

$$\|l\| \geq \|\xi\| - \delta \geq \bar{\Delta}(4M + \Lambda)^{-1} - \delta$$
$$\geq 2^{-1}\bar{\Delta}(4M + \Lambda)^{-1}. \tag{8.33}$$

In view of (8.33),

$$l \neq 0.$$

Let

$$u \in B(x - f_j(x)\|l\|^{-2}l, \delta). \tag{8.34}$$

It follows from (8.22), (8.30), and (8.34) that

$$\|u - y\| \leq \delta + \|x - f_j(x)\|l\|^{-2}l - y\|$$
$$\leq \delta + \|f_j(x)\|\xi\|^{-2}\xi - f_j(x)\|l\|^{-2}l\|$$
$$\leq \delta + M_0\| \|\xi\||^{-2}\xi - \|l\|^{-2}l\|. \tag{8.35}$$

In view of (8.18), (8.21), (8.32), and (8.33),

$$\| \|\xi\|^{-2}\xi - \|l\|^{-2}l\| \leq \|l\|^{-2}\|l - \xi\| + \|\xi\| | \|\xi\|^{-2} - \|l\|^{-2}|$$
$$\leq \|l\|^{-2}\|l - \xi\| + \|l\|^{-2}\|\xi\|^{-1}| \|l\|^2 - \|\xi\|^2|$$
$$\leq \|l\|^{-2}[\delta + \|\xi\|^{-1}\delta(\|l\| + \|\xi\|)]$$
$$\leq \|l\|^{-2}\delta[1 + \|\xi\|^{-1}(2\|\xi\| + \delta)] \leq 4\delta\|l\|^{-2}$$
$$\leq 16\delta(4M + \Lambda)^2\bar{\Delta}^{-2}. \tag{8.36}$$

Relations (8.35) and (8.36) imply that

$$\|u - y\| \leq \delta + 16\delta M_0\bar{\Delta}^{-2}(4M + \Lambda)^2$$
$$= \delta(1 + 16M_0\bar{\Delta}^{-2}(4M + \Lambda)^2). \tag{8.37}$$

It follows from (8.29) and (8.37) that

$$\|u - z\| \le \|u - y\| + \|y - z\|$$
$$\le \|z - x\| + \delta(1 + 16M_0\bar{\Delta}^{-2}(4M + \Lambda)^2). \tag{8.38}$$

This completes the proof of Lemma 8.4. □

8.4 Proof of Theorem 8.3

In view of (8.7), there exists

$$z \in B(0, M) \cap C. \tag{8.39}$$

Relations (8.14) and (8.39) imply that

$$\|x_0 - z\| \le 2M. \tag{8.40}$$

Set

$$\epsilon_0 = 0. \tag{8.41}$$

We show that for all integers $i \ge 0$,

$$\|z - x_i\| \le 2M + \Delta_1 \sum_{j=0}^{i} \epsilon_j. \tag{8.42}$$

By (8.40) and (8.41), inequality (8.42) holds for $i = 0$.

Assume that $i \ge 0$ is an integer and (8.42) holds. It follows from (8.12), (8.39), (8.41), and (8.42) that

$$\|x_i\| \le 3M + \Delta_1 \sum_{j=0}^{i} \epsilon_j$$
$$\le 3M + \Delta_1 \sum_{j=0}^{\infty} \epsilon_j \le 3M + \Lambda. \tag{8.43}$$

If $f_{S(i)}(x_i) \le \Delta$, then by (8.15) and (8.42),

$$x_{i+1} = x_i,$$

$$\|z - x_{i+1}\| \le 2M + \Delta_1 \sum_{j=0}^{i+1} \epsilon_j.$$

Assume that

$$f_{S(i)}(x_i) > \Delta.$$

In view of (8.10), (8.11), (8.16), (8.17), (8.39), (8.42), (8.43), and Lemma 8.4 applied with $x = x_i$, $j = S(i)$, $\delta = \epsilon_{i+1}$, $\bar{\Delta} = \Delta$, $u = x_{k+1}$, $\xi = l_k$,

$$\|x_{k+1} - z\| \le \|x_k - z\| + \epsilon_{i+1}\Delta_1 \le 2M + \Delta_1 \sum_{j=0}^{i+1} \epsilon_j.$$

Thus (8.42) holds for all integers $i \ge 0$.

By (8.12), (8.39), and (8.42), for all integers $i \ge 0$,

$$\|z - x_i\| \le 2M + \Lambda,$$
$$\|x_i\| \le 3M + \Lambda. \tag{8.44}$$

Let $i \ge 0$ be an integer. It follows from (8.10), (8.39), (8.43), and Lemma 8.4 applied with $x = x_i$, $j = S(i)$, $\delta = \epsilon_{i+1}$, $\bar{\Delta} = \Delta$, $u = x_{i+1}$, $\xi = l_i$ that the following property holds:

(P2) if

$$f_{S(i)}(x_i) > \Delta,$$

then

$$\|x_{i+1} - z\| \le \|x_i - z\| + \epsilon_{i+1}\Delta_1$$

and there exists $y_i \in X$ such that

$$\|x_{i+1} - y_i\| \le \epsilon_{i+1}\Delta_1, \tag{8.45}$$
$$\|y_i - z\|^2 \le \|x_i - z\|^2 - \|x_i - y_i\|^2. \tag{8.46}$$

Assume that

$$f_{S(i)}(x_i) > \Delta. \tag{8.47}$$

Property (P2) and (8.47) imply that there exists $y_i \in X$ satisfying (8.45) and (8.46). By (8.39), (8.44), and (8.46),

$$\|y_i - z\| \le \|x_i - z\| \le 2M + \Lambda, \tag{8.48}$$

$$\|y_i\| \le 3M + \Lambda. \tag{8.49}$$

It follows from (8.44), (8.45), and (8.48) that

$$\begin{aligned} &|\|x_{i+1} - z\|^2 - \|y_i - z\|^2| \\ &\le |\|x_{i+1} - z\| - \|y_i - z\||(\|x_{i+1} - z\| + \|y_i - z\|) \\ &\le 2(2M + \Lambda)\|x_{i+1} - y_i\| \le 2\epsilon_{i+1}\Delta_1(2M + \Lambda). \end{aligned} \tag{8.50}$$

By (8.44), (8.45), and (8.48),

$$\begin{aligned} &|\|x_i - y_i\|^2 - \|x_i - x_{i+1}\|^2| \\ &\le |\|x_i - y_i\| - \|x_i - x_{i+1}\||(\|x_i - y_i\| + \|x_i - x_{i+1}\|) \\ &\le \|x_{i+1} - y_i\|(\|x_i - z\| + \|z - y_i\| + \|x_i - z\| + \|z - x_{i+1}\|) \\ &\le 4(\epsilon_i\Delta_1 + 1)(2M + \Lambda). \end{aligned} \tag{8.51}$$

It follows from (8.46), (8.50), and (8.51) that

$$\begin{aligned} &\|x_{i+1} - z\|^2 \le \|y_i - z\|^2 + 2\epsilon_{i+1}\Delta_1(2M + \Lambda) \\ &\le \|z - x_i\|^2 - \|x_i - y_i\|^2 + 2\epsilon_{i+1}\Delta_1(2M + \Lambda) \\ &\le \|z - x_i\|^2 - \|x_i - x_{i+1}\|^2 \\ &+4\epsilon_{i+1}\Delta_1(2M + \Lambda) + 2\epsilon_{i+1}\Delta_1(2M + \Lambda) \\ &= \|z - x_i\|^2 - \|x_i - x_{i+1}\|^2 + 6\epsilon_{i+1}\Delta_1(2M + \Lambda). \end{aligned}$$

Thus we have shown that the following property holds:

(P3) if

$$f_{S(i)}(x_i) > \Delta,$$

then

$$\|x_{i+1} - z\|^2 \le \|z - x_i\|^2 - \|x_i - x_{i+1}\|^2 + 6\epsilon_{i+1}\Delta_1(2M + \Lambda).$$

Assume that an integer $N \ge 0$ and that

$$\|x_{n+1} - x_n\| \le \gamma_0 \text{ for } n = N, \dots, N + \bar{p} - 1. \tag{8.52}$$

This implies that for all $n_1, n_2 \in \{N, \dots, N + \bar{p}\}$,

$$\|x_{n_1} - x_{n_2}\| \le \bar{p}\gamma_0. \tag{8.53}$$

Let $i \in \{1, \dots, m\}$. By (P1), there is $q \in \{N, \dots, N + \bar{p} - 1\}$ such that

$$S(q) = i. \tag{8.54}$$

We show that

$$f_i(x_q) = f_{S(q)}(x_q) \le \Delta_1. \tag{8.55}$$

Assume the contrary. Then

$$f_{S(q)}(x_q) > \Delta. \tag{8.56}$$

By (8.17), (8.52), and (8.56),

$$\begin{aligned} \gamma_0 &\ge \|x_{q+1} - x_q\| \\ &\|f_{S(q)}(x_q)\|l_q\|^{-2}l_q\| - \epsilon_{q+1}, \\ \gamma_0 + \epsilon_{q+1} &> \Delta\|l_q\|^{-2}\|l_q\| > \Delta\|l_q\|^{-1}. \end{aligned} \tag{8.57}$$

By (8.16) and (8.56), there exists

$$\xi_q \in \partial f_{S(q)}(x_q) \tag{8.58}$$

such that

$$\|\xi_q - l_q\| < \epsilon_{q+1}. \tag{8.59}$$

In view of (8.9), (8.44), and (8.58),

$$\|\xi_q\| \le M_1 - 2. \tag{8.60}$$

In view of (8.11), (8.59), and (8.60),

$$\|l_q\| \le M_1 - 1. \tag{8.61}$$

It follows from (8.11), (8.57), and (8.61) that

$$\gamma_0 + \epsilon_{q+1} > \Delta(M_1 - 1)^{-1}$$

and

$$\gamma_0 > \Delta(M_1 - 1)^{-1} - \epsilon_{q+1}$$
$$\geq M_1^{-1}\Delta - 2^{-1}M_1^{-1}\Delta(4M + \Lambda)^{-1} \geq 2^{-1}\Delta M_1^{-1}.$$

This inequality contradicts (8.13). This contradiction we have reached proves (8.55).

Let $n \in \{N, \dots, N + \bar{p}\}$. It follows from (8.9), (8.44), and (8.53)–(8.55) that

$$f_i(x_n) \leq f_{S(q)}(x_q) + |f_{S(q)}(x_n) - f_{S(q)}(x_q)|$$
$$\leq \Delta + (M_1 - 2)\|x_n - x_q\|$$
$$\leq \Delta + \bar{p}\gamma_0(M_1 - 2),$$
$$f_i(x_n) \leq \Delta + \bar{p}\gamma_0(M_1 - 2) < 2\Delta$$
$$\text{for } n = N, \dots, N + \bar{p} \text{ and all integers } i = 1, \dots, m.$$

Thus we have shown that the following property holds:

(P4) if an integer $N \geq 0$ and (8.52) holds, then

$$f_i(x_n) < 2\Delta$$
$$\text{for } n = N, \dots, N + \bar{p} \text{ and all integers } i = 1, \dots, m.$$

Set

$$E_0 = \{n \in \{0, 1, \dots, \} : \|x_n - x_{n+1}\| > \gamma_0\}, \tag{8.62}$$
$$E_1 = \{n \in \{0, 1, \dots\} : \{n, \dots, n + \bar{p} - 1\} \cap E_0 \neq \emptyset\}. \tag{8.63}$$

Property (P3), (8.12), (8.15), and (8.40) imply that for any natural number n,

$$4M^2 \geq \|z - x_0\|^2$$
$$\geq \|z - x_0\|^2 - \|z - x_n\|^2$$
$$= \sum_{i=0}^{n-1}[\|z - x_i\|^2 - \|z - x_{i+1}\|^2]$$
$$\geq \sum_{i=0}^{n-1}[\|x_i - x_{i+1}\|^2 - 6(2M + \Lambda)\Delta_1\epsilon_{i+1}]$$
$$\geq \sum_{i=0}^{n-1}\|x_i - x_{i+1}\|^2 - 6(2M + \Lambda)\Lambda$$

and

$$4M^2 + 6(2M + \Lambda)\Lambda$$
$$\geq \sum_{i=0}^{n-1} \|x_i - x_{i+1}\|^2$$
$$\geq \gamma_0^2 \mathrm{Card}(\{i \in \{0, \dots, n-1\} : \|x_i - x_{i+1}\| > \gamma_0\}),$$
$$\mathrm{Card}(\{i \in \{0, \dots, n-1\} : \|x_i - x_{i+1}\| > \gamma_0\}) \geq \gamma_0^{-2}(4M^2 + 6(2M + \Lambda)\Lambda).$$

Since the inequality above holds for any natural number n, we conclude that

$$\mathrm{Card}(E_0) = \mathrm{Card}(\{i \in \{0, \dots, n-1\} : \|x_i - x_{i+1}\| > \gamma_0\})$$
$$\leq \gamma_0^{-2}(4M^2 + 6(2M + \Lambda)\Lambda).$$

By the relation above and (8.63),

$$\mathrm{Card}(E_1) \leq \bar{p}\, \mathrm{Card}(E_0)$$
$$\leq \bar{p}\gamma_0^{-2}(4M^2 + 6(2M + \Lambda)\Lambda).$$

Together with (8.62), (8.63), and property (P4) this completes the proof of Theorem 8.3. □

8.5 Dynamic String-Averaging Subgradient Projection Algorithm

In this chapter we study convergence of dynamic string-averaging subgradient projection algorithms for solving convex feasibility problems in a general Hilbert space. Our goal is to obtain an approximate solution of the problem in the presence of perturbations. We show that our subgradient projection algorithm generates a good approximate solution, if the perturbations are summable.

Let us now describe the convex feasibility problem studied in the chapter and dynamic string-averaging subgradient projection algorithms which will be used for its solving.

Let m be a natural number and $f_i : X \to R^1$, $i = 1, \dots, m$ be convex continuous functions.

For each $i = 1, \dots, m$ set

$$C_i = \{x \in X : f_i(x) \leq 0\},$$
$$C = \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : f_i(x) \leq 0\}.$$

Suppose that

$$C \neq \emptyset.$$

Recall that a point $x \in C$ is called a solution of our feasibility problem. For a given $\epsilon > 0$ a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if

$$f_i(x) \leq \epsilon \text{ for all } i = 1, \dots, m.$$

In this chapter we apply a dynamic string-averaging subgradient projection method with variable strings and weights in order to obtain a good approximative solution of the feasibility problem.

Next we describe the dynamic string-averaging subgradient method with variable strings and weights.

By an index vector, we mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q. \tag{8.64}$$

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{t \in \Omega} w(t) = 1. \tag{8.65}$$

Fix a number

$$\bar{\Delta} \in (0, m^{-1}] \tag{8.66}$$

and an integer

$$\bar{q} \geq m. \tag{8.67}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \leq \bar{q} \text{ for all } t \in \Omega, \tag{8.68}$$

$$w(t) \geq \bar{\Delta} \text{ for all } t \in \Omega. \tag{8.69}$$

For each $x \in X$, each $\epsilon \geq 0$, each $\bar{\epsilon} \geq 0$, and each $i \in \{1, \dots, m\}$ set

$$A_i(x, \bar{\epsilon}, \epsilon) := \{x\} \text{ if } f_i(x) \leq \bar{\epsilon} \tag{8.70}$$

and if $f_i(x) > \bar{\epsilon}$, then set

$$A_i(x, \bar{\epsilon}, \epsilon) = \{x - f_i(x)\|l\|^{-2}l : \ l \in \partial f_i(x) + B(0, \epsilon), \ l \neq 0\} + B(0, \epsilon). \quad (8.71)$$

Let $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector, $\epsilon \geq 0$, $\bar{\epsilon} \geq 0$. Define

$$A_0(t, x, \bar{\epsilon}, \epsilon) = \{(y, \lambda) \in X \times R^1 : \ \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that}$$
$$y_0 = x, \quad (8.72)$$

for each $i = 1, \dots, p(t)$,

$$y_i \in A_{t_i}(y_{i-1}, \bar{\epsilon}, \epsilon), \quad (8.73)$$
$$y = y_{p(t)}, \quad (8.74)$$
$$\lambda = \max\{\|y_i - y_{i-1}\| : \ i = 1, \dots, p(t)\}\}. \quad (8.75)$$

Let $x \in X$, $(\Omega, w) \in \mathcal{M}$, $\epsilon \geq 0$, $\bar{\epsilon} \geq 0$. Define

$$A(x, (\Omega, w), \bar{\epsilon}, \epsilon) = \{(y, \lambda) \in X \times R^1 : \ \text{there exist}$$
$$(y_t, \lambda_t) \in A_0(t, x, \bar{\epsilon}, \epsilon), \ t \in \Omega \quad (8.76)$$

such that

$$\|y - \sum_{t \in \Omega} w(t) y_t\| \leq \epsilon, \quad (8.77)$$
$$\lambda = \max\{\lambda_t : \ t \in \Omega\}\}. \quad (8.78)$$

Fix a natural number $\bar{N}$.
Suppose that $M > 1$, $M_0 > 0$, $M_1 > 2$, $\Lambda > 0$, and $\Delta \in (0, 1]$ be such that

$$B(0, M) \cap \{x \in X : \ f_i(x) \leq 0, \ i = 1, \dots, m\} \neq \emptyset, \quad (8.79)$$
$$f_i(B(0, 3M + \Lambda)) \subset [-M_0, M_0], \ i = 1, \dots, m, \quad (8.80)$$
$$|f_i(u) - f_i(v)| \leq (M_1 - 2)\|u - v\|$$
$$\text{for all } u, v \in B(0, 3M + \Lambda + 1) \text{ and all } i = 1, \dots, m. \quad (8.81)$$

Let

$$\Delta_0 = 2^{-1} \Delta (4M + \Lambda)^{-1} M_1^{-1} \bar{N}^{-1}, \quad (8.82)$$
$$\Delta_1 = 1 + 16 M_0 \Delta^{-2} (4M + \Lambda)^2, \quad (8.83)$$
$$\gamma_0 = 4^{-1} \Delta (\bar{N} + 1)^{-1} M_1^{-1} \bar{q}^{-1}. \quad (8.84)$$

Theorem 8.5 *Assume that*

$$\{\epsilon_i\}_{i=1}^{\infty} \subset (0, \Delta_0], \tag{8.85}$$

$$\sum_{i=1}^{\infty} \epsilon_i \le (\Delta_1 \bar{q} + 1)^{-1} \Lambda, \tag{8.86}$$

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{8.87}$$

satisfies for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}), \tag{8.88}$$

$$x_0 \in B(0, M), \tag{8.89}$$

$$\{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{8.90}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \Delta, \epsilon_i). \tag{8.91}$$

Then

$$\|x_i\| \le 3M + \Lambda \text{ for all integers } i \ge 0,$$

$$Card(\{n \in 0, 1, \dots\} : \max\{\lambda_{i+1} : i = n, \dots, n + \bar{N} - 1\} > \gamma_0\})$$
$$\le \bar{\Delta}^{-1} \bar{N} \gamma_0^{-2} (2M + \Lambda)(2M + 7\Lambda).$$

Moreover, if an integer $n \ge 0$ satisfies

$$\lambda_i \le \gamma_0, \ i = n, \dots, n + \bar{N} - 1,$$

then, for all integers $k = n, \dots, n + \bar{N}$ and each $s = 1, \dots, m$, $f_s(x_k) < 2\Delta$.

8.6 Proof of Theorem 8.5

By (8.79), there exists

$$z \in B(0, M) \cap C. \tag{8.92}$$

In view of (8.89) and (8.92),

$$\|x_0 - z\| \le 2M. \tag{8.93}$$

Set

$$\epsilon_0 = 0. \tag{8.94}$$

Let n be a natural number. In view of (8.91),

$$(x_n, \lambda_n) \in A(x_{n-1}, (\Omega_n, w_n), \Delta, \epsilon_n). \tag{8.95}$$

By (8.76)–(8.78) and (8.95), there exist

$$(y_{n,t}, \lambda_{n,t}) \in A_0(t, x_{n-1}, \Delta, \epsilon_n),\ t \in \Omega_n \tag{8.96}$$

such that

$$\|x_n - \sum_{t \in \Omega_n} w_n(t) y_{n,t}\| \le \epsilon_n, \tag{8.97}$$

$$\lambda_n = \max\{\lambda_{n,t} :\ t \in \Omega_n\}. \tag{8.98}$$

It follows from (8.72)–(8.75) and (8.96) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_n$ there exists a sequence $\{y_{n,t,i}\}_{i=0}^{p(t)} \subset X$ such that

$$y_{n,t,0} = x_{n-1}, \tag{8.99}$$

$$y_{n,t,i} \in A_{t_i}(y_{n,t,i-1}, \Delta, \epsilon_n),\ i = 1, \dots, p(t), \tag{8.100}$$

$$y_{n,t} = y_{n,t,p(t)}, \tag{8.101}$$

$$\lambda_{n,t} = \max\{\|y_{n,t,i} - y_{n,t,i-1}\| :\ i = 1, \dots, p(t)\}. \tag{8.102}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_n.$$

Assume that an integer

$$j \in \{0, \dots, p(t) - 1\}.$$

In view of (8.100),

$$y_{n,t,j+1} \in A_{t_{j+1}}(y_{n,t,j}, \Delta, \epsilon_n). \tag{8.103}$$

If

$$f_{t_{j+1}}(y_{n,t,j}) \le \Delta,$$

then (8.70) and (8.103) imply that

$$y_{n,t,j+1} = y_{n,t,j}. \tag{8.104}$$

Assume that

$$f_{t_{j+1}}(y_{n,t,j}) > \Delta \tag{8.105}$$

and

$$\|y_{n,t,j}\| \le 3M + \Lambda. \tag{8.106}$$

It follows from (8.71), (8.82), (8.83), (8.85), (8.92), (8.103), (8.105), (8.106), and Lemma 8.4 applied with $\delta = \epsilon_n$, $\Delta = \bar{\Delta}$, $x = y_{n,t,j}$, $u = y_{n,t,j+1}$ that

$$\begin{aligned}\|y_{n,t,j+1} - z\| &\le \|y_{n,t,j} - z\| + \epsilon_n(1 + 16M_0\Delta^{-2}(4M + \Lambda)^2)\\ &\le \|y_{n,t,j} - z\| + \epsilon_n\Delta_1\end{aligned}$$

and that there exists

$$\tilde{y}_{n,t,j} \in X$$

such that

$$\begin{aligned}&\|\tilde{y}_{n,t,j} - y_{n,t,j+1}\|\\ &\le \epsilon_n(1 + 16M_0\Delta^{-2}(4M + \Lambda)^2) \le \epsilon_n\Delta_1\end{aligned}$$

and

$$\|\tilde{y}_{n,t,j} - z\|^2 \le \|y_{n,t,j} - z\|^2 - \|y_{n,t,j} - \tilde{y}_{n,t,j}\|^2.$$

Thus we have shown that for each natural number n, each

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_n$$

and each $j \in \{0, \dots, p(t) - 1\}$, the following properties hold:

(P5) if

$$f_{t_{j+1}}(y_{n,t,j}) \le \Delta,$$

then

$$y_{n,t,j+1} = y_{n,t,j};$$

(P6) if

$$f_{t_{j+1}}(y_{n,t,j}) > \Delta$$

and

$$\|y_{n,t,j}\| \le 3M + \Lambda,$$

then

$$\|y_{n,t,j+1} - z\| \le |y_{n,t,j} - z\| + \epsilon_n \Delta_1 \tag{8.107}$$

and there exists

$$\tilde{y}_{n,t,j} \in X$$

such that

$$\|\tilde{y}_{n,t,j} - y_{n,t,j+1}\| \le \epsilon_n \Delta_1 \tag{8.108}$$

and

$$\|\tilde{y}_{n,t,j} - z\|^2 \le \|y_{n,t,j} - z\|^2 - \|y_{n,t,j} - \tilde{y}_{n,t,j}\|^2. \tag{8.109}$$

Assume that $n \ge 0$ is an integer and that

$$\|z - x_n\| \le 2M + (\Delta_1 \bar{q} + 1) \sum_{i=0}^{n} \epsilon_i. \tag{8.110}$$

(Note that in view of (8.93) and (8.94), inequality (8.110) holds for $n = 0$.)

Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{n+1}$. By induction we show that for all $j = 0, \dots, p(t)$,

$$\|z - y_{n+1,t,j}\| \le 2M + (\Delta_1 \bar{q} + 1) \sum_{i=0}^{n} \epsilon_i + j\epsilon_{n+1}\Delta_1. \tag{8.111}$$

In view of (8.99) and (8.110) inequality (8.111) holds for $j = 0$.

Assume that $j \in \{0, \dots, p(t) - 1\}$ and (8.111) holds. It follows from (8.86), (8.92), and (8.111) that

$$\|z - y_{n+1,t,j}\| \le 2M + \Lambda, \ \|y_{n+1,t,j}\| \le 3M + \Lambda. \tag{8.112}$$

Properties (P5) and (P6), (8.111), and (8.112) imply that

$$\|y_{n+1,t,j+1} - z\| \le \|y_{n+1,t,j} - z\| + \epsilon_{n+1}\Delta_1$$
$$\le 2M + (\Delta_1\bar{q} + 1)\sum_{i=0}^{n}\epsilon_i + (j+1)\epsilon_{n+1}\Delta_1.$$

Therefore (8.111) holds for all $j = 0, \dots, p(t)$ and in view of (8.68) and (8.101),

$$\|y_{n+1,t} - z\| \le 2M + (\Delta_1\bar{q} + 1)\sum_{i=0}^{n}\epsilon_i + \bar{q}\epsilon_{n+1}\Delta_1. \tag{8.113}$$

By the convexity of the norm, (8.65), (8.97), and (8.113),

$$\|z - x_{n+1}\|$$
$$\le \|z - \sum_{t\in\Omega_{n+1}} w_{n+1}(t)y_{n+1,t}\| + \|\sum_{t\in\Omega_{n+1}} w_{n+1}(t)y_{n+1,t} - x_{n+1}\|$$
$$\le \sum_{t\in\Omega_{n+1}} w_{n+1}(t)\|z - y_{n+1,t}\| + \epsilon_{n+1}$$
$$\le 2M + (\Delta_1\bar{q} + 1)\sum_{i=0}^{n}\epsilon_i + (\bar{q}\Delta_1 + 1)\epsilon_{n+1}.$$

Thus we have shown by induction that the following property holds:

(P7) for all integers $n \ge 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{n+1}$ and all $j = 0, \dots, p(t)$, relations (8.110), (8.111), and (8.113) hold.

Property (P7), (8.86), (8.92), (8.110), (8.111), and (8.113) imply that for all integers $n \ge 0$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{n+1}$ and all $j = 0, \dots, p(t)$,

$$\|z - x_n\| \le 2M + \Lambda,\ \|x_n\| \le 3M + \Lambda, \tag{8.114}$$

$$\|z - y_{n+1,t,j}\| \le 2M + \Lambda,\ \|y_{n+1,t,j}\| \le 3M + \Lambda, \tag{8.115}$$

$$\|z - y_{n+1,t}\| \le 2M + \Lambda,\ \|y_{n+1,t}\| \le 3M + \Lambda, \tag{8.116}$$

Let $n \ge 0$ be an integer, $t = (t_1, \dots, t_{p(t)}) \in \Omega_{n+1}$ and $j \in \{0, \dots, p(t) - 1\}$. Assume that

$$f_{t_{j+1}}(y_{n+1,t,j}) > \Delta. \tag{8.117}$$

Property (P6), (8.115), and (8.117) imply that there exists

$$\tilde{y}_{n+1,t,j} \in X$$

such that

$$\|\tilde{y}_{n+1,t,j} - y_{n+1,t,j+1}\| \le \epsilon_{n+1}\Delta_1 \tag{8.118}$$

and

$$\|\tilde{y}_{n+1,t,j} - z\|^2 \le \|y_{n+1,t,j} - z\|^2 - \|y_{n+1,t,j} - \tilde{y}_{n+1,t,j}\|^2. \tag{8.119}$$

In view of (8.92), (8.115), and (8.119),

$$\|z - \tilde{y}_{n+1,t,j}\| \le 2M + \Lambda,\ \ \|\tilde{y}_{n+1,t,j}\| \le 3M + \Lambda. \tag{8.120}$$

By (8.115), (8.118), and (8.120),

$$\begin{aligned}
&|\|y_{n+1,t,j+1} - z\|^2 - \|\tilde{y}_{n+1,t,j} - z\|^2| \\
&\le |\|y_{n+1,t,j+1} - z\| - \|\tilde{y}_{n+1,t,j} - z\||(\|y_{n+1,t,j+1} - z\| + \|\tilde{y}_{n+1,t,j} - z\|) \\
&\le 2(2M + \Lambda)\|y_{n+1,t,j+1} - \tilde{y}_{n+1,t,j}\| \\
&\le 2(2M + \Lambda)\epsilon_{n+1}\Delta_1.
\end{aligned} \tag{8.121}$$

By (8.115), (8.118), and (8.120),

$$\begin{aligned}
&|\|y_{n+1,t,j} - y_{n+1,t,j+1}\|^2 - \|y_{n+1,t,j} - \tilde{y}_{n+1,t,j}\|^2| \\
&\le |\|y_{n+1,t,j} - y_{n+1,t,j+1}\| - \|y_{n+1,t,j} - \tilde{y}_{n+1,t,j}\|| \\
&\times(\|y_{n+1,t,j} - y_{n+1,t,j+1}\| + \|y_{n+1,t,j} - \tilde{y}_{n+1,t,j}\|) \\
&\le 4(2M + \Lambda)\|y_{n+1,t,j+1} - \tilde{y}_{n+1,t,j}\| \\
&\le 4(2M + \Lambda)\epsilon_{n+1}\Delta_1.
\end{aligned} \tag{8.122}$$

It follows from (8.119), (8.121), and (8.122) that

$$\begin{aligned}
&\|y_{n+1,t,j+1} - z\|^2 \le \|\tilde{y}_{n+1,t,j} - z\|^2 + 2(2M + \Lambda)\epsilon_{n+1}\Delta_1 \\
&\le \|y_{n+1,t,j} - z\|^2 - \|y_{n+1,t,j} - \tilde{y}_{n+1,t,j}\|^2 + 2(2M + \Lambda)\epsilon_{n+1}\Delta_1 \\
&\le \|y_{n+1,t,j} - z\|^2 - \|y_{n+1,t,j} - y_{n+1,t,j+1}\|^2 + 6(2M + \Lambda)\epsilon_{n+1}\Delta_1.
\end{aligned}$$

Thus if

$$f_{t_{j+1}}(y_{n+1,t,j}) > \Delta,$$

then

$$\begin{aligned}
&\|y_{n+1,t,j+1} - z\|^2 \\
&\le \|y_{n+1,t,j} - z\|^2 - \|y_{n+1,t,j} - y_{n+1,t,j+1}\|^2 + 6(2M + \Lambda)\epsilon_{n+1}\Delta_1.
\end{aligned}$$

Together with property (P5) this implies that for each integer $n \geq 0$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{n+1}$ and each $j \in \{0, \dots, p(t) - 1\}$,

$$\|y_{n+1,t,j+1} - z\|^2 \\ \leq \|y_{n+1,t,j} - z\|^2 - \|y_{n+1,t,j} - y_{n+1,t,j+1}\|^2 + 6(2M + \Lambda)\epsilon_{n+1}\Delta_1. \quad (8.123)$$

Set

$$E_0 = \{n \in \{0, 1, \dots\} : \lambda_{n+1} > \gamma_0\}. \quad (8.124)$$

Let

$$n \in E_0. \quad (8.125)$$

By (8.98), (8.102), (8.124), and (8.125), there exist $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \Omega_{n+1}$ and $s \in \{0, \dots, p(\tau) - 1\}$ such that

$$\gamma_0 < \lambda_{n+1} = \lambda_{n+1,\tau} = \|y_{n+1,\tau,s} - y_{n+1,\tau,s+1}\|. \quad (8.126)$$

In view of (8.123) and (8.126),

$$\|y_{n+1,\tau,s+1} - z\|^2 \leq \|y_{n+1,\tau,s} - z\|^2 - \gamma_0^2 + 6(2M + \Lambda)\epsilon_{n+1}\Delta_1. \quad (8.127)$$

It follows from (8.68), (8.99), (8.101), (8.123), and (8.127) that

$$\begin{aligned}
&\|x_n - z\|^2 - \|y_{n+1,\tau} - z\|^2 \\
&= \|y_{n+1,\tau,0} - z\|^2 - \|y_{n+1,\tau,p(\tau)} - z\|^2 \\
&= \sum_{j=0}^{p(\tau)-1} [\|y_{n+1,\tau,j} - z\|^2 - \|y_{n+1,\tau,j+1} - z\|^2] \\
&\geq \gamma_0^2 - 6(2M + \Lambda)\epsilon_{n+1}\Delta_1 p(\tau) \\
&\geq \gamma_0^2 - 6(2M + \Lambda)\epsilon_{n+1}\Delta_1 \bar{q}. \quad (8.128)
\end{aligned}$$

By (8.68), (8.99), (8.101), and (8.123), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{n+1}$,

$$\begin{aligned}
&\|x_n - z\|^2 - \|y_{n+1,t} - z\|^2 \\
&= \|y_{n+1,t,0} - z\|^2 - \|y_{n+1,t,p(t)} - z\|^2 \\
&= \sum_{j=0}^{p(t)-1} [\|y_{n+1,t,j} - z\|^2 - \|y_{n+1,t,j+1} - z\|^2] \\
&\geq -6(2M + \Lambda)\epsilon_{n+1}\Delta_1 \bar{q}. \quad (8.129)
\end{aligned}$$

By convexity of the norm, (8.65), (8.97), (8.114), and (8.116),

$$|\|x_{n+1}-z\|^2-\|\sum_{t\in\Omega_{n+1}}w_{n+1}(t)y_{n+1,t}-z\|^2|$$

$$\leq |\|x_{n+1}-z\|-\|\sum_{t\in\Omega_{n+1}}w_{n+1}(t)y_{n+1,t}-z\||$$

$$\times(\|x_{n+1}-z\|+\|\sum_{t\in\Omega_{n+1}}w_{n+1}(t)y_{n+1,t}-z\|)$$

$$\leq \|x_{n+1}-\sum_{t\in\Omega_{n+1}}w_{n+1}(t)y_{n+1,t}\|(2M+\Lambda+\sum_{t\in\Omega_{n+1}}w_{n+1}(t)\|y_{n+1,t}-z\|)$$

$$\leq 2\epsilon_{n+1}(2M+\Lambda). \tag{8.130}$$

It follows from the convexity of the function $\|\cdot\|^2$, (8.65), (8.69), (8.128), and (8.129) that

$$\|\sum_{t\in\Omega_{n+1}}w_{n+1}(t)y_{n+1,t}-z\|^2$$

$$\leq \sum_{t\in\Omega_{n+1}}w_{n+1}(t)\|y_{n+1,t}-z\|^2$$

$$=w_{n+1}(\tau)\|y_{n+1,\tau}-z\|^2$$

$$+\sum\{w_{n+1}(t)\|y_{n+1,t}-z\|^2:\ t\in\Omega_{n+1}\setminus\{\tau\}\}$$

$$\leq w_{n+1}(\tau)(\|x_n-z\|^2-\gamma_0^2+6(2M+\Lambda)\epsilon_{n+1}\Delta_1\bar{q})$$

$$+\sum\{w_{n+1}(t)(\|x_n-z\|^2+6(2M+\Lambda)\epsilon_{n+1}\Delta_1\bar{q}):\ t\in\Omega_{n+1}\setminus\{\tau\}\}$$

$$\leq \|x_n-z\|^2+6(2M+\Lambda)\epsilon_{n+1}\Delta_1\bar{q}-\bar{\Delta}\gamma_0^2. \tag{8.131}$$

In view of (8.91), (8.124), (8.125), (8.130), and (8.131),

$$\|x_{n+1}-z\|^2\leq \|x_n-z\|^2-\bar{\Delta}\gamma_0^2+2(2M+\Lambda)\epsilon_{n+1}(3\bar{q}\Delta_1+1) \tag{8.132}$$

for every $n\in E_0$.

Let $n\geq 0$ be an integer. By (8.68), (8.99), (8.101), and (8.123), for each $t=(t_1,\dots,t_{p(t)})\in\Omega_{n+1}$,

$$\|x_n-z\|^2-\|y_{n+1,t}-z\|^2$$

$$=\|y_{n+1,t,0}-z\|^2-\|y_{n+1,t,p(t)}-z\|^2$$

$$= \sum_{j=0}^{p(t)-1} [\|y_{n+1,t,j} - z\|^2 - \|y_{n+1,t,j+1} - z\|^2]$$

$$\geq -6(2M + \Lambda)\epsilon_{n+1}\Delta_1\bar{q}. \quad (8.133)$$

By the convexity of the function $\|\cdot\|^2$, (8.65), and (8.133),

$$\|\sum_{t\in\Omega_{n+1}} w_{n+1}(t)y_{n+1,t} - z\|^2$$

$$\leq \sum_{t\in\Omega_{n+1}} w_{n+1}(t)\|y_{n+1,t} - z\|^2$$

$$\leq \sum_{t\in\Omega_{n+1}} w_{n+1}(t)[\|x_n - z\|^2 + +6(2M + \Lambda)\epsilon_{n+1}\Delta_1\bar{q}]$$

$$= \|x_n - z\|^2 + 6(2M + \Lambda)\epsilon_{n+1}\Delta_1\bar{q}. \quad (8.134)$$

In view of the convexity of the norm, (8.97), (8.114), and (8.116),

$$|\|x_{n+1} - z\|^2 - \|\sum_{t\in\Omega_{n+1}} w_{n+1}(t)y_{n+1,t} - z\|^2|$$

$$\leq \|x_{n+1} - \sum_{t\in\Omega_{n+1}} w_{n+1}(t)y_{n+1,t}\|$$

$$\times(\|x_{n+1} - z\| + \|\sum_{t\in\Omega_{n+1}} w_{n+1}(t)y_{n+1,t} - z\|)$$

$$\leq \epsilon_{n+1}(4M + 2\Lambda).$$

Together with (8.134) this implies that

$$\|x_{n+1} - z\|^2$$

$$\leq \|x_n - z\|^2 + 6(2M + \Lambda)\epsilon_{n+1}\Delta_1\bar{q} + \epsilon_{n+1}(4M + 2\Lambda)$$

$$= \|x_n - z\|^2 + 2(2M + \Lambda)\epsilon_{n+1}(3\Delta_1\bar{q} + 1). \quad (8.135)$$

Let n be a natural number. By (8.86), (8.93), (8.132), and (8.135),

$$(2M + \Lambda)^2 \geq \|x_0 - z\|^2$$

$$\geq \|x_0 - z\| - \|x_n - z\|^2$$

$$= \sum_{k=0}^{n-1} (\|x_k - z\|^2 - \|x_{k+1} - z\|^2)$$

$$= \sum \{\|x_k - z\|^2 - \|x_{k+1} - z\|^2 : \ k \in E_0 \cap [0, n-1]\}$$

$$+ \sum \{\|x_k - z\|^2 - \|x_{k+1} - z\|^2 : \ k \in \{0, \dots, n-1\} \setminus E_0\}$$

$$\geq \sum \{\bar{\Delta}\gamma_0^2 - 2(2M + \Lambda)\epsilon_{k+1}(3\bar{q}\Delta_1 + 1) : \ k \in E_0 \cap [0, n-1]\}$$

$$+ \sum \{-2(2M + \Lambda)\epsilon_{k+1}(3\Delta_1\bar{q} + 1) : \ k \in \{0, \dots, n-1\} \setminus E_0\}$$

$$= \bar{\Delta}\gamma_0^2 \mathrm{Card}(E_0 \cap [0, n-1]) - (\sum_{k=0}^{n-1} \epsilon_{k+1})(4M + 2\Lambda)(3\bar{q}\Delta_1 + 1)$$

$$\geq \bar{\Delta}\gamma_0^2 \mathrm{Card}(E_0 \cap [0, n-1]) - (\sum_{i=1}^{\infty} \epsilon_i)(4M + 2\Lambda)(3\bar{q}\Delta_1 + 1)$$

and

$$\mathrm{Card}(E_0 \cap [0, n-1])$$
$$\leq \bar{\Delta}^{-1}\gamma_0^{-2}((2M + \Lambda)^2 + (4M + 2\Lambda)(3\bar{q}\Delta_1 + 1)\Lambda(\Delta_1\bar{q} + 1)^{-1})$$
$$\leq \bar{\Delta}^{-1}\gamma_0^{-2}(2M + \Lambda)(2M + 7\Lambda).$$

Since the relation above holds for any natural number n we conclude that

$$\mathrm{Card}(E_0) \leq \bar{\Delta}^{-1}\gamma_0^{-2}(2M + \Lambda)(2M + 7\Lambda). \tag{8.136}$$

Set

$$E_1 = \{n \in \{0, 1, \dots, \} : \ \{n, \dots, n + \bar{N} - 1\} \cap E_0 \neq \emptyset\}. \tag{8.137}$$

In view of (8.136) and (8.137),

$$\mathrm{Card}(E_1) \leq \bar{N}\mathrm{Card}(E_0) \leq \bar{N}\bar{\Delta}^{-1}\gamma_0^{-2}(2M + \Lambda)(2M + 7\Lambda). \tag{8.138}$$

Assume that an integer $n \geq 0$ satisfies

$$n \notin E_1. \tag{8.139}$$

By (8.124), (8.137), and (8.139),

$$\{n, \dots, n + \bar{N} - 1\} \cap E_0 = \emptyset,$$
$$\lambda_{k+1} \leq \gamma_0, \ k = n, \dots, n + \bar{N} - 1. \tag{8.140}$$

It follows from the convexity of the function $\|\cdot\|$, (8.65), (8.68), (8.85), (8.97)–(8.99), (8.101), (8.102), and (8.140) that for each integer $k \in \{n, \dots, n + \bar{N} - 1\}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$, and each $j \in \{0, \dots, p(t) - 1\}$,

$$\gamma_0 \ge \lambda_{k+1} \ge \lambda_{k+1,t} \ge \|y_{k+1,t,j+1} - y_{k+1,t,j}\|, \tag{8.141}$$

$$\|x_k - y_{k+1,t,j}\| \le j\gamma_0,$$

$$\|x_k - y_{k+1,t,j+1}\| \le (j+1)\gamma_0, \tag{8.142}$$

$$\|x_k - y_{k+1,t}\| \le \bar{q}\gamma_0, \tag{8.143}$$

$$\begin{aligned}
&\|x_k - x_{k+1}\| \\
&\le \|x_k - \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t}\| \\
&+ \|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t} - x_{k+1}\| \\
&\le \sum_{t \in \Omega_{k+1}} w_{k+1}(t) \|x_k - y_{k+1,t}\| + \epsilon_{k+1} \\
&\le \bar{q}\gamma_0 + \epsilon_{k+1} \le \bar{q}\gamma_0 + \Delta_0.
\end{aligned} \tag{8.144}$$

In view of (8.144), for each $k_1, k_2 \in \{n, \dots, n + \bar{N}\}$,

$$\|x_{k_1} - x_{k_2}\| \le \bar{N}(\bar{q}\gamma_0 + \Delta_0). \tag{8.145}$$

Let $s \in \{1, \dots, m\}$. By (8.88) there exist

$$k \in \{n, \dots, n + \bar{N} - 1\},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1},\ j \in \{0, \dots, p(t) - 1\} \tag{8.146}$$

such that

$$s = t_{j+1}. \tag{8.147}$$

It follows from (8.103) and (8.146) that

$$y_{k+1,t,j+1} \in A_{t_{j+1}}(y_{k+1,t,j}, \Delta, \epsilon_{k+1}). \tag{8.148}$$

Assume that

$$f_{t_{j+1}}(y_{k+1,t,j}) > \Delta. \tag{8.149}$$

By (8.71), (8.82), (8.84), (8.85), (8.141), (8.148), and (8.149),

$$\gamma_0 \ge \|y_{k+1,t,j+1} - y_{k+1,t,j}\| \ge f_{t_{j+1}}(y_{k+1,t,j}) - \epsilon_{k+1},$$

$$f_{t_{j+1}}(y_{k+1,t,j}) \le \gamma_0 + \Delta_0 < \Delta.$$

This contradicts (8.149). The contradiction we have reached proves that

$$\Delta \geq f_{t_{j+1}}(y_{k+1,t,j}) = f_s(y_{k+1,t,j}). \tag{8.150}$$

By (8.68), (8.81), (8.114), (8.115), (8.142), and (8.150),

$$\begin{aligned} f_s(x_k) &\leq f_s(y_{k+1,t,j}) + |f_s(x_k) - f_s(y_{k+1,t,j})| \\ &\leq \Delta + (M_1 - 2)\|x_k - y_{k+1,t,j}\| \\ &\leq \Delta + (M_1 - 2)\bar{q}\gamma_0. \end{aligned} \tag{8.151}$$

It follows from (8.82), (8.84), (8.114), (8.145), and (8.151) that for each $p \in \{n, \dots, n + \bar{N} - 1\}$,

$$\begin{aligned} f_s(x_p) &\leq f_s(x_k) + |f_s(x_p) - f_s(x_k)| \\ &\leq \Delta + (M_1 - 2)\bar{q}\gamma_0 + (M_1 - 2)\|x_p - x_k\| \\ &\leq \Delta + (M_1 - 2)\bar{q}\gamma_0 + (M_1 - 2)\bar{N}(\bar{q}\gamma_0 + \Delta_0) \\ &\leq \Delta + (M_1 - 2)[\bar{q}\gamma_0(\bar{N} + 1) + \bar{N}\gamma_0) \leq 2\Delta \end{aligned}$$

for all $s = 1, \dots, m$. Theorem 8.5 is proved.

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